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NULLING IN LINEAR ARRAY PATTERNS WITH MINIMIZATION OF WEIGHT PERTURBATIONS

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Robert A. Shore
Hans Steyskal

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ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, NY 13441

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APPROVED: Philipp Blacksmith

PHILIPP BLACKSMITH
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APPROVED: Allan C. Schell

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An analysis is performed of nulling in linear array patterns subject to minimization of the weight perturbation. Both combined amplitude and phase perturbation, and phase-only perturbation are treated. The solutions obtained are interpreted in terms of cancellation beams superimposed on the initial pattern.

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1. INTRODUCTION

Adaptive antenna systems presently receive much interest and there has been considerable study of adaptive algorithms. Less attention has been devoted to the end product of the algorithms, which is an antenna pattern with nulls in the directions of the jammers. In this report we therefore study the general properties of patterns with null constraints. The hope is that this may contribute to more insight into the basic behavior and fundamental limitations of adaptive nulling systems and possibly also suggest new adaptive schemes suitable for large array antennas.

The approach we take is based on a direct synthesis of the array pattern with the desired nulls, in contrast to an iterative solution using an adaptive algorithm. The problem is formulated as an approximation problem: a set of complex array weights $\{w_n\}$ is determined which, subject to the null constraints, best approximates a given set of weights $\{w_{on}\}$. Two types of approximation criteria are considered: (a) minimum relative weight perturbations; that is, $\sum |(w_n - w_{on})/w_{on}|^2 = \min.$, and (b) minimum total weight perturbations; that is, $\sum |w_n - w_{on}|^2 = \min.$ Criteria related to (b) have been applied before,[1,2] whereas

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1. Mayhan, J. (1976) Nulling limitations for a multiple-beam antenna, IEEE Trans. Antennas Propag. AP-24:769-779.
2. Steyskal, H. (1982) Synthesis of antenna patterns with nulls, IEEE Trans. Antennas Propag. AP-30.

the criterion (a) seems to be new. For both cases exact solutions are derived and interpreted in terms of cancellation beams superimposed on the initial, unconstrained pattern.

We then consider the same synthesis problem under the restriction that the array excitation is perturbed in phase only. The motivation for this is, of course, that in a phased array the required electronic controls are available at no extra cost. This subject is of great practical interest, and several publications on it have appeared in the literature.[3-9] The associated approximation problem is nonlinear in general but can be linearized by assuming that the required phase perturbations are small, an assumption which is reasonable in certain applications involving null placement in low sidelobe regions. Based on this assumption we determine a set of phase perturbations $\{\phi_n\}$ for the same two criteria as above which now take the forms (a) minimum phase perturbations; that is, $\sum\phi_n^2$ = min., and (b) minimum products of the phase perturbations with the element amplitudes; that is, $\sum(|w_{on}|\phi_n)^2$ = min. This time the solutions can be interpreted as pairs of cancellation beams superimposed on the initial pattern.

Following the analysis of these null synthesis problems we present the results of computations designed to display the basic features of the solutions obtained and to explore the limitations of the small phase perturbation assumption used to linearize the phase-only null synthesis problem.

We conclude the report by demonstrating an equivalence between the null synthesis problem treated in terms of obtaining a best fit to a given set of element weights, and treated in terms of obtaining a best approximation to the original pattern. At the same time we also extend the null synthesis problems studied in the first part of the report by considering a more general minimization criterion that includes the criteria treated earlier as special cases.

2. ANALYSIS

In the following analysis we consider a linear array of equispaced isotropic elements (see Figure 1). The spacing between the elements is d and the phase reference center is taken to be the center of the array. Letting w_n, n = 1, 2 ..., N, be the complex weight of the n^{th} array element, the array field pattern, p(u) is given by

(Due to the large number of references cited above, they will not be listed here. See References, page 67.)

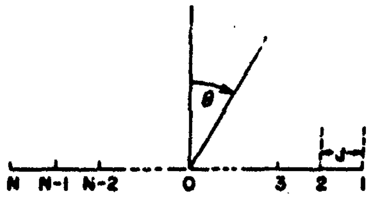


Figure 1. Geometry of Array

$$p(u) = \sum_{n=1}^{N} w_n e^{-jd_n u}$$

where

$$d_n = \frac{N-1}{2} - (n-1), \qquad n = 1, 2, \ldots, N$$

and

$$u = kd \sin\theta$$

with

$$k = \frac{2\pi}{\lambda}$$

and θ the angle measured from broadside to the array.* Note that the d_n are odd symmetric with respect to the phase reference center; that is,

$$d_n = -d_{N-n+1}, \qquad n = 1, 2, \ldots, N .$$

The general problem we address here is the following. Suppose we are given an amplitude taper a_n, $n = 1, 2, \ldots, N$, for the element excitations (for example, for low sidelobes), and a direction u_s for the peak of the array pattern (look direction). Then the array coefficients are

$$w_{on} = a_n e^{jd_n u_s}, \qquad n = 1, 2, \ldots, N.$$

The amplitude distribution is assumed to be symmetrical with respect to the array center. We now wish to alter the element weights so as to place nulls in the array pattern at a set of M prescribed sidelobe locations u_k, $k = 1, 2, \ldots, M$. What should the coefficient perturbations be? We consider two principal cases of this problem: (1) the perturbations are of both amplitude and phase; and (2) the perturbations are of the phases only.

*In this report, we use the implicit time dependence $e^{-j\omega t}$.

2.1 Amplitude and Phase Perturbations

We begin by considering the case in which both the amplitude and the phase of the element weights can be perturbed from their original values. The perturbed coefficients can be represented as

$$w_n = a_n e^{jd_n u_s} + a_n e^{jd_n u_s} (\Delta_n + j\,\phi_n) . \tag{1}$$

The first term on the right hand side of Eq. (1) is the initial value, w_{on}, of the n^{th} element weight, and the second term is the perturbation of the weight

$$\Delta w_n = a_n e^{jd_n u_s} (\Delta_n + j\,\phi_n) . \tag{2}$$

The array field pattern is then

$$\begin{aligned} p(u) &= \sum_{n=1}^{N} \left[a_n e^{jd_n u_s} + a_n e^{jd_n u_s} (\Delta_n + j\,\phi_n) \right] e^{-jd_n u} \\ &= p_o(u) + \sum_{n=1}^{N} a_n e^{jd_n u_s} (\Delta_n + j\,\phi_n)\, e^{-jd_n u} \end{aligned}$$

where $p_o(u)$ is the original or unperturbed array pattern. We now wish to determine the Δ_n and the ϕ_n to place nulls at the M locations u_k, $k = 1, 2, \ldots, M$, or equivalently to find solutions to the equation system

$$\sum_{n=1}^{N} a_n (\Delta_n + j\,\phi_n)\, e^{jd_n u_s}\, e^{jd_n u_k} = - p_o(u_k), \; k = 1, 2, \ldots, M . \tag{3}$$

Note that in this equation system there are 2N unknowns and 2M equations [considering the real and imaginary parts of Eq. (3) separately]. Hence, if the number, M, of prescribed null locations is less than N, the equation system as it stands does not have a uniquely determined solution. Clearly a further requirement (or requirements) must be imposed on the solution to determine it uniquely.

At this point, motivated by the desire to keep the weight perturbations as small as possible, two possibilities suggest themselves. The first is to find the solution to Eq. (3) which minimizes the sum of the squares of the absolute values of the weight perturbations relative to the original weights.

$$\sum_{n=1}^{N} |\Delta_n + j\,\phi_n|^2 = \sum_{n=1}^{N} (\Delta_n^2 + \phi_n^2) .$$

The second possibility is to find the solution which minimizes the sum of the squares of the absolute values of the total weight perturbations

$$\sum_{n=1}^{N} |\Delta w_n|^2 = \sum_{n=1}^{N} a_n^2 (\Delta_n^2 + \phi_n^2) .$$

We now treat each of these two cases in turn. The treatment of both cases is based on the following result which we state as a theorem:

THEOREM A:

Let A be an m × n complex matrix of rank m, m < n (that is, A has m linearly independent rows). Then the solution $\underline{x}$ of the equation system

$$A\underline{x} = \underline{y}$$

which minimizes the norm of $\underline{x}$, $||\underline{x}|| = (\underline{x}^\dagger \underline{x})^{1/2}$, can be expressed as a linear combination of the columns of $A^\dagger$. (Here † represents the Hermitian transpose operator.)

PROOF:

The proof is an immediate corollary of the fact that the minimum norm solution is[10]

$$\underline{x} = A^\dagger (AA^\dagger)^{-1} \underline{y} .$$

If we let $\underline{b}$ be the m × 1 vector $(AA^\dagger)^{-1} \underline{y}$ then

$$\underline{x} = A^\dagger \underline{b}$$

so that $\underline{x}$ is a linear combination of the columns of $A^\dagger$ with coefficients from $\underline{b}$.

Applying this theorem first to finding the solution of Eq. (3) which minimizes $\sum_{n=1}^{N} (\Delta_n^2 + \phi_n^2)$ we let

$$\underline{x} = [\Delta_1 + j\,\phi_1, \Delta_2 + j\,\phi_2, \ldots, \Delta_N + j\,\phi_N]^T \tag{4}$$

$$\underline{y} = -[p_o(u_1), p_o(u_2), \ldots, p_o(u_M)]^T \tag{5}$$

and

10. Rao, C., and Mitra, S. (1971) Generalized Inverse of Matrices and Its Applications, John Wiley & Sons, New York, p 48.

$$
A = \begin{bmatrix} a_1 e^{jd_1(u_s - u_1)} & a_2 e^{jd_2(u_s - u_1)} & \cdots & a_N e^{jd_N(u_s - u_1)} \\ a_1 e^{jd_1(u_s - u_2)} & a_2 e^{jd_2(u_s - u_2)} & \cdots & a_N e^{jd_N(u_s - u_2)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_1 e^{jd_1(u_s - u_M)} & a_2 e^{jd_2(u_s - u_M)} & \cdots & a_N e^{jd_N(u_s - u_M)} \end{bmatrix} \tag{6}
$$

so that the system of Eqs. (3) can be written as $A\underline{x} = \underline{y}$. It follows from the theorem that

$$
\Delta_n + j\phi_n = \sum_{m=1}^{M} b_m a_n e^{jd_n(u_m - u_s)}, \quad n = 1, 2, \ldots, N \tag{7}
$$

where the vector of coefficients, $\underline{b}$, is given by inverting the equation

$$
(AA^\dagger)\underline{b} = \underline{y}. \tag{8}
$$

The elements of the $M \times M$ matrix $AA^\dagger$ are given by

$$
\begin{aligned}
[AA^\dagger]_{km} &= \sum_{n=1}^{N} a_n e^{jd_n(u_s - u_k)} a_n e^{-jd_n(u_s - u_m)} \\
&= \sum_{n=1}^{N} a_n^2 e^{-jd_n(u_k - u_m)} \\
&= \sum_{n=1}^{N} a_n^2 \cos[d_n(u_k - u_m)], \qquad k, m = 1, 2, \ldots, M
\end{aligned}
$$

making use of the odd symmetry of the d_n and the assumed even symmetry of the original distribution of the element weight amplitudes. The elements of $AA^\dagger$ are thus real. The components of the vector $\underline{y}$, the negatives of the unperturbed pattern values at the prescribed null locations, are likewise real since the unperturbed pattern is real because of the symmetry of the d_n and of the original amplitude distribution. Hence the coefficients b_m, $m = 1, 2, \ldots M$, are real and it follows from Eq. (7) that

$$\Delta_n = a_n \sum_{m=1}^{M} b_m \cos[d_n(u_m - u_s)]$$

$$\phi_n = a_n \sum_{m=1}^{M} b_m \sin[d_n(u_m - u_s)] \ .$$

The Δ_n are thus even symmetric and the ϕ_n odd symmetric with respect to n: that is,

$$\left.\begin{aligned} \Delta_n &= \Delta_{N-n+1} \\ \phi_n &= -\phi_{N-n+1} \end{aligned}\right\} \qquad n = 1, 2, \ldots, N \ .$$

The representation of the relative weight perturbation in the form of Eq. (7) can be given an appealing physical interpretation. Substituting Eq. (7) into Eq. (2) we obtain

$$\Delta w_n = a_n^2 \sum_{m=1}^{M} b_m e^{j d_n u_m}$$

and thus

$$p(u) = p_o(u) + \Delta p(u)$$

where

$$\Delta p(u) = \sum_{m=1}^{M} b_m \sum_{n=1}^{N} a_n^2 e^{-j d_n (u - u_m)} \ . \qquad (9)$$

Accordingly the perturbed pattern equals the original pattern plus the sum of M cancellation beams each corresponding to a taper of the amplitude of the element excitations proportional to the square of the original amplitude taper and with a pattern peak in the direction of one of the desired null locations. In the sequel we will sometimes refer to the cancellation beams corresponding to the amplitude taper $a_n{}^2$ as "tapered cancellation beams."

If instead of minimizing $\sum_{n=1}^{N} (\Delta_n^2 + \phi_n^2)$ we minimize $\sum_{n=1}^{N} |\Delta w_n|^2 = \sum_{n=1}^{N} a_n^2(\Delta_n^2 + \phi_n^2)$ (see p. 13), then similarly to Eq. (4) and Eq. (6) we let

$$\underline{x} = [a_1(\Delta_1 + j\phi_1),\ a_2(\Delta_2 + j\phi_2),\ \ldots,\ a_N(\Delta_N + j\phi_N)]^T$$

and

$$A = \begin{bmatrix} e^{jd_1(u_s - u_1)} & e^{jd_2(u_s - u_1)} & \ldots & e^{jd_N(u_s - u_1)} \\ e^{jd_1(u_s - u_2)} & e^{jd_2(u_s - u_2)} & \ldots & e^{jd_N(u_s - u_2)} \\ \vdots & \vdots & & \vdots \\ e^{jd_1(u_s - u_M)} & e^{jd_2(u_s - u_M)} & \ldots & e^{jd_N(u_s - u_M)} \end{bmatrix}$$

thereby obtaining

$$a_n(\Delta_n + j\phi_n) = \sum_{m=1}^{M} b_m\, e^{jd_n(u_m - u_s)},\quad n = 1, 2, \ldots, N .$$

The vector of coefficients, $\underline{b}$, is obtained as before by inverting Eq. (8) where the elements of the matrix $AA^\dagger$ are now given by

$$\left[AA^\dagger\right]_{km} = \sum_{n=1}^{N} \cos[d_n(u_k - u_m)],\ k, m = 1, 2, \ldots, M .$$

The coefficients b_m, $m = 1, 2, \ldots, M$ are again real and we obtain

$$\Delta_n = \frac{1}{a_n} \sum_{m=1}^{M} b_m \cos[d_n(u_m - u_s)]$$

$$\phi_n = \frac{1}{a_n} \sum_{m=1}^{M} b_m \sin[d_n(u_m - u_s)] .$$

The weight perturbation takes the form

$$\Delta w_n = \sum_{m=1}^{M} b_m\, e^{jd_n u_m},\quad n = 1, 2, \ldots, N \tag{10}$$

and the pattern perturbation is

$$\Delta p(u) = \sum_{m=1}^{M} b_m \sum_{n=1}^{N} e^{-jd_n(u-u_m)} \tag{11a}$$

$$= \sum_{m=1}^{M} b_m \frac{\sin[N(u-u_m)/2]}{\sin[(u-u_m)/2]} . \tag{11b}$$

The perturbed pattern is thus equal to the sum of the original pattern and M cancellation beams each corresponding to a uniform amplitude distribution of the element excitations and with a peak in the direction of one of the null locations. We will sometimes refer to such cancellation beams as "uniform cancellation beams."

The representation of the total pattern perturbation as the sum of M cancellation beams corresponding to uniform element amplitudes is also the form of the pattern perturbations for a half-wavelength-spacing linear array obtained by Steyskal[2] who finds the weights that will give the pattern with nulls at a set of prescribed locations which differs as little as possible in the mean square sense from the original pattern. In Appendix A we give an alternate derivation of this result as a special case of the solution to the minimum pattern perturbation problem for a linear array with arbitrary uniform element spacing. Thus the solution to the problem of placing nulls at a prescribed set of locations which minimizes the sum of the squares of the absolute values of the total weight perturbations $\sum_{n=1}^{N} |\Delta w_n|^2 = \sum_{n=1}^{N} a_n^2 (\Delta_n^2 + \phi_n^2)$ is also the solution that minimizes the mean square difference between the original field pattern and the perturbed field pattern.

In low sidelobe applications we can anticipate that the minimum (in either of the two cases we have treated above) weight perturbations required to place nulls at a set of sidelobe locations will be small. Starting with the expression for the n^{th} perturbed weight

$$w_n = a_n e^{jd_n u_s} (1 + \Delta_n + j\phi_n)$$

$$= a_n e^{jd_n u_s} [(1+\Delta_n)^2 + \phi_n^2]^{1/2} e^{j\tan^{-1}\left(\frac{\phi_n}{1+\Delta_n}\right)}$$

we can then expand

$$\left[(1+\Delta_n)^2 + \phi_n^2\right]^{1/2} = (1+\Delta_n)\left[1 + \frac{1}{2}\left(\frac{\phi_n}{1+\Delta_n}\right)^2 + \ldots\right]$$

$$\tan^{-1}\left(\frac{\phi_n}{1+\Delta n}\right) = \phi_n - \phi_n\Delta_n + \ldots$$

and retain only the linear terms thus obtaining

$$w_n \approx a_n (1 + \Delta_n)\, e^{j d_n u_s}\, e^{j\phi_n}\,.$$

Hence for small Δ_n and ϕ_n, Δ_n is the relative change in the amplitude and ϕ_n is the phase perturbation of the n^{th} weight.

2.2 Phase-Only Perturbations

We now turn our attention to the case in which phase perturbations only are allowed in the element weights. The amplitudes of the element excitations are fixed at their original values. The phase-only analysis parallels that performed above in which both amplitude and phase were allowed to vary from their initial values.

The perturbed coefficients are represented by

$$w_n = a_n\, e^{j d_n u_s}\, e^{j\phi_n} \tag{12a}$$

$$= a_n\, e^{j d_n u_s} + a_n\, e^{j d_n u_s} \left(e^{j\phi_n} - 1\right), \quad n = 1, 2, \ldots, N\,. \tag{12b}$$

We assume that the phase perturbations are small (for example, for applications to null synthesis in low sidelobe array patterns). Then

$$e^{j\phi_n} - 1 \approx j\phi_n \tag{13}$$

and

$$w_n \approx a_n\, e^{j d_n u_s} + j\, a_n \phi_n\, e^{j d_n u_s}\,. \tag{14}$$

The array pattern is given by

$$p(u) = p_o(u) + j \sum_{n=1}^{N} a_n \phi_n\, e^{j d_n u_s}\, e^{-j d_n u}$$

and the equation system for the nulls is

$$\left.\begin{aligned} &\sum_{n=1}^{N} a_n \phi_n \sin\left[d_n (u_k - u_s)\right] = -\, p_o(u_k) \qquad (15a)\\ &\sum_{n=1}^{N} a_n \phi_n \cos\left[d_n (u_k - u_s)\right] = 0 \qquad (15b)\end{aligned}\right\}\quad k = 1, 2, \ldots, M\,.$$

This equation system has 2 M equations for N unknowns and hence, as in the case of perturbations of both amplitude and phase, a unique solution to the system of equations can be obtained only if a further requirement is imposed on the solution. (It is assumed of course that the number of unknown phase perturbations is greater than twice the number of null locations.) As above we treat two cases, the first here corresponding to the requirement that the solution minimize the sum of the squares of the phase perturbations, $\sum_{n=1}^{N} \phi_n^2$ (or equivalently, minimize the sum of the absolute values of the weight perturbations relative to the original weights), and the second to the requirement that the solution minimize the sum of the squares of the absolute values of the total weight perturbations, $\sum_{n=1}^{N} (a_n \phi_n)^2$.

Consider first minimizing the sum of the squares of the phase perturbations. We write the system of Eqs. (15) in the form

$$A\underline{x} = \underline{y} \tag{16}$$

where

$$\underline{x} = [\phi_1, \phi_2, \ldots, \phi_N]^T \tag{17}$$

$$A = \begin{bmatrix} a_1 \sin[d_1(u_1 - u_s)] & a_2 \sin[d_2(u_1 - u_s)] & \cdots & a_N \sin[d_N(u_1 - u_s)] \\ a_1 \sin[d_1(u_2 - u_s)] & a_2 \sin[d_2(u_2 - u_s)] & \cdots & a_N \sin[d_N(u_2 - u_s)] \\ \vdots & \vdots & & \vdots \\ a_1 \sin[d_1(u_M - u_s)] & a_2 \sin[d_2(u_M - u_s)] & \cdots & a_N \sin[d_N(u_M - u_s)] \end{bmatrix} \tag{18}$$

and $\underline{y}$ is given by Eq. (5). Using Theorem A we then obtain the representation of the phase perturbation as

$$\phi_n = a_n \sum_{m=1}^{M} b_m \sin[d_n(u_m - u_s)] \, . \tag{19}$$

The vector of coefficients, $\underline{b}$, is obtained by inverting the equation

$$(AA^T)\, \underline{b} = \underline{y} \tag{20}$$

with the elements of the matrix AA^T given by

$$\left[AA^T\right]_{km} = \sum_{n=1}^{N} a_n^2 \sin[d_n(u_k - u_s)] \sin[d_n(u_m - u_s)] . \tag{21}$$

It follows from Eq. (19), because of the even symmetry of the amplitudes a_n assumed throughout and the odd symmetry of the d_n, that the ϕ_n are odd symmetric with respect to the phase reference center; that is,

$$\phi_n = -\phi_{N-n+1}, \quad n = 1, 2, \ldots, N .$$

This means that Eq. (15b) is automatically satisfied by the solution obtained to Eq. (15a).

It is important to note that the values of ϕ_n calculated from Eq. (19) must be substituted in Eq. (12a) or Eq. (12b) to obtain the perturbed weights. Since the values for the ϕ_n were obtained using the approximation (14), for the weights, this means that the resulting pattern nulls cannot be expected to be theoretically perfect as they would be if the calculated values of ϕ_n were substituted in approximation (14).*

The approximated form (14) <u>is</u> useful, however, for obtaining a physical interpretation of the phase-only perturbation solution. Substituting Eq. (19) in (14) we obtain the approximate weight perturbation

$$\begin{aligned}
\Delta w_n &= w_n - a_n e^{jd_n u_s} \\
&\approx j\, a_n \phi_n e^{jd_n u_s} \\
&= j a_n^2 \sum_{m=1}^{M} b_m e^{jd_n u_s} \sin[d_n(u_m - u_s)] \\
&= \frac{a_n^2}{2} \sum_{m=1}^{M} b_m \left[e^{jd_n u_m} - e^{jd_n(2u_s - u_m)} \right]
\end{aligned}$$

*The value of the perturbed pattern at a prescribed null location u_k is in fact given by

$$\sum_{n=1}^{N} a_n \left[e^{j\phi_n} - (1 + j\phi_n) \right] e^{-jd_n u_k} \approx \sum_{n=1}^{N} a_n \frac{\phi_n^2}{2} e^{-jd_n u_k}$$

with the value of the power pattern at the location given approximately by

$$\frac{1}{4} \sum_{m=1}^{N} \sum_{n=1}^{N} a_m a_n \phi_m^2 \phi_n^2 e^{-j(d_n - d_m)u_k} .$$

and the corresponding pattern perturbation

$$\Delta p(u) = \sum_{m=1}^{M} b_m \sum_{n=1}^{N} \frac{a_n^2}{2} \left[e^{-jd_n(u-u_m)} - e^{-jd_n\left(u - \left\{2u_s - u_m\right\}\right)} \right] \quad (22)$$

The perturbed pattern is thus approximately equal to the sum of the original pattern and the sum of the differences of M pairs of cancellation beams. Each of these pairs of beams corresponds to an element amplitude taper proportional to the square of the original amplitude taper, with one member of the pair having a peak in the direction of a null location and the other member of the beam pair having a peak in the symmetric direction on the other side of the original pattern look direction, u_s. [Note that $2u_s - u_m = u_m - 2(u_m - u_s)$.] Thus the placing of nulls in the prescribed locations is accompanied by a doubling of the pattern at symmetric locations on the other side of the look direction.

The requirement of minimizing the sum of the squares of the absolute values of the total weight perturbations $\sum_{n=1}^{N} (a_n \phi_n)^2$, (see p. 19) leads, similarly to Eq. (17) and Eq. (18), first to expressing the system of Eqs. (15a) in the form of Eq. (16) where now

$$\underline{x} = [a_1\phi_1, a_2\phi_2, \ldots, a_N\phi_N]^T$$

and

$$A = \begin{bmatrix} \sin[d_1(u_1 - u_s)] & \sin[d_2(u_1 - u_s)] & \cdots & \sin[d_N(u_1 - u_s)] \\ \sin[d_1(u_2 - u_s)] & \sin[d_2(u_2 - u_s)] & \cdots & \sin[d_N(u_2 - u_s)] \\ \vdots & \vdots & & \\ \sin[d_1(u_M - u_s)] & \sin[d_2(u_M - u_s)] & \cdots & \sin[d_N(u_M - u_s)] \end{bmatrix}.$$

Hence, similarly to Eq. (19),

$$a_n\phi_n = \sum_{m=1}^{M} b_m \sin[d_n(u_m - u_s)] \quad (23)$$

with the vector of coefficients, $\underline{b}$, obtained by inverting Eq. (20) with

$$\left[AA^T\right]_{km} = \sum_{n=1}^{N} \sin[d_n(u_k - u_s)] \sin[d_n(u_m - u_s)] .$$

The weight perturbation is then approximated by

$$\Delta w_n \approx \frac{1}{2} \sum_{m=1}^{M} b_m \left[e^{j d_n u_m} - e^{j d_n (2u_s - u_m)} \right]$$

and the corresponding pattern perturbation by

$$\Delta p(u) \approx \sum_{m=1}^{M} \frac{b_m}{2} \left[e^{-j d_n (u - u_m)} - e^{-j d_n (u - \{2u_s - u_m\})} \right] \quad (24a)$$

$$= \sum_{m=1}^{M} \frac{b_m}{2} \left\{ \frac{\sin[N(u - u_m)/2]}{\sin[(u - u_m)/2]} - \frac{\sin[N(u - \{2u_s - u_m\})/2]}{\sin[(u - \{2u_s - u_m\})/2]} \right\} . \quad (24b)$$

Minimizing the squares of the total weight perturbations thus leads to a result similar to that for minimizing the squares of the phase perturbations (relative weight perturbations). The perturbing or sidelobe cancelling pattern is again approximated by the sum of the differences of M pairs of beams, each of which corresponds here to a uniform amplitude distribution of the element weights. Equation (23) shows that the phase perturbations are inversely proportional to the amplitudes of the weights, whereas for the minimum-phase-perturbation solution, Eq. (19), the phase perturbations are directly proportional to the weight amplitudes. It is also simple to show (see Appendix A) that for half-wavelength element spacing, the pattern given by the approximation (24) which corresponds to minimizing the approximate total weight perturbations, is also the pattern that differs least from the original pattern in a mean square sense, while having nulls at the desired locations obtained with small phase-only perturbations.

It is instructive to compare the approximate (for small phase-only perturbations) representations of the pattern perturbations obtained here, (22) and (24), with the representations in Eqs. (9) and (11), obtained earlier when both the amplitude and phase of the weights are allowed to vary. Minimizing the sum of the squares of the absolute values of the relative (to the original amplitude weight) perturbations ($e^{j\phi_n} - 1 \approx j\phi_n$ for phase-only perturbations and $\Delta_n + j\phi_n$ for combined amplitude and phase perturbations) leads to a representation of the cancellation pattern in terms of beams corresponding to an element amplitude taper proportional to the square of the original amplitudes; whereas minimizing the sum of the squares of the absolute values of the total weight perturbations ($a_n e^{ju_s} (e^{j\phi_n} - 1) \approx j a_n e^{ju_s}$, $a_n e^{ju_s} (\Delta_n + j\phi_n)$, respectively) leads to a representation of the cancellation pattern in terms of beams corresponding to a uniform amplitude distribution. Phase-only perturbations require two beams for each desired null, one pointing in the direction of the null and the other, of opposite sign, in the symmetric direction on the opposite side of the original main beam axis, with the result that the placement of the nulls is achieved at the expense of doubling the sidelobe levels at the symmetric locations. When both amplitude and phase are allowed to vary, there is only one beam for each null pointing in the direction of the null location. For half-wavelength element spacing, the patterns corresponding to minimizing the sum of the squares of the total weight perturbations are also the patterns that differ least in the mean square sense from the original patterns. Because of the small phase approximation (13) employed in the phase-only analysis, the resulting pattern nulls are not perfect, whereas the patterns achieved with both amplitude and phase control have theoretically perfect nulls at the desired locations.

Before discussing numerical results, it is worthwhile to introduce a useful normalization of the beam coefficients. Consider first Eqs. (9) and (11) obtained for the perturbing or sidelobe cancelling pattern $\Delta p(u)$ when both the amplitude and the phase of the weights are allowed to vary from their original values. For simplicity suppose there is only one null location (that is, $M = 1$). Then Eq. (9) gives for minimum squared relative weight perturbations,

$$-p_o(u_1) = \Delta p(u_1) = b_1 \sum_{n=1}^{N} a_n^2$$

so that

$$b_1 = -\frac{p_o(u_1)}{\sum_{n=1}^{N} a_n^2} \quad .$$

whereas Eq. (11) gives, for minimum squared total weight perturbations,

$$-p_o(u_1) = \Delta p(u_1) = N b_1$$

or

$$b_1 = -\frac{p_o(u_1)}{N} \, .$$

The beam coefficients in the two cases can have widely differing values depending on the amplitude taper, a_n, and the number of elements in the array, N. If simultaneous nulls at multiple locations are desired, then the deviations of the values of the beam coefficients from their values obtained for the same null locations taken one at a time give a measure of the extent of coupling or interference between the component beams of the perturbing pattern. For ease in describing the extent of coupling between component beams as well as for comparing differences in coupling corresponding to different types of beams, it is desirable to normalize the beam coefficients so that their value equals the negative of the unperturbed pattern value at the null location when there is only one null location. This is equivalent to requiring that each component beam of the cancelling pattern have a magnitude of unity in the peak of direction of the beam. It follows that the beam coefficients for "tapered cancellation beams" should be multiplied by $\sum_{n=1}^{N} a_n^2$, and the beam coefficients for "uniform cancellation beams" should be multiplied by N. The two cases can be given a uniform representation as follows. Let

$$c_n = \begin{cases} a_n, & \text{tapered cancellation beams} \\ 1, & \text{uniform cancellation beams} \end{cases} \tag{25}$$

and

$$t_n = \frac{c_n^2}{\sum_{n=1}^{N} c_n^2} . \tag{26}$$

Then the equation system for the normalized beam coefficients, b_m, is [see Eq. (8)]

$$(AA^{\dagger})\, \underline{b}' = \underline{v}$$

where

$$\left[AA^{\dagger}\right]_{km} = \sum_{n=1}^{N} t_n \cos[d_n(u_k - u_m)], \quad k, m = 1, 2, \ldots, M .$$

The n^{th} weight perturbation is given by

$$\Delta w_n = t_n \sum_{n=1}^{N} b'_m\, e^{jd_n u_m}$$

and the cancellation pattern by

$$\Delta p(u) = \sum_{m=1}^{M} b_m' \sum_{n=1}^{N} t_n \, e^{-jd_n(u - u_m)} .$$

The real and imaginary parts of the relative weight perturbations are given respectively by

$$\Delta_n = \frac{t_n}{a_n} \sum_{m=1}^{M} b_m' \cos[d_n(u_m - u_s)]$$

$$\phi_n = \frac{t_n}{a_n} \sum_{m=1}^{M} b_m' \sin[d_n(u_m - u_s)] .$$

The normalization for phase-only control is entirely similar. With t_n defined by Eq. (26), the equation system for the normalized beam coefficients is [cf. Eq. (20)]

$$(AA^T)\, \underline{b}' = \underline{y}$$

where

$$\left[AA^T\right]_{km} = \sum_{n=1}^{N} 2t_n \sin[d_n(u_k - u_s)] \sin[d_n(u_m - u_s)] .$$

The phase perturbations are

$$\phi_n = 2\frac{t_n}{a_n} \sum_{m=1}^{M} b_m' \sin[d_n(u_m - u_s)] .$$

The perturbation of the n^{th} weight can be represented approximately by

$$\Delta w_n \approx t_n \sum_{m=1}^{M} b_m' \left[e^{jd_n u_m} - e^{jd_n(2u_s - u_m)}\right]$$

and the cancellation pattern approximated by

$$\Delta p(u) \approx \sum_{m=1}^{M} b_m' \sum_{n=1}^{N} t_n \left[e^{-jd_n(u - u_m)} - e^{-jd_n(u - \{2u_s - u_m\})}\right].$$

3. NUMERICAL RESULTS AND DISCUSSIONS

In this section we present and discuss the results of computations performed to display the basic features of the solutions obtained in the previous section. All computations were done for an array with 41 isotropic elements with half wavelength spacing. Unless otherwise stated, the original or unperturbed pattern corresponds to a 40 dB Chebyshev taper of the element excitations.[11]

We begin by discussing the simplest case—that of imposing a single null in the pattern. A location of $\theta = 15.23°$ was chosen corresponding to the approximate location of the peak of the fifth sidelobe of the unperturbed pattern. Figures 2a-2c and 3a-3c show the patterns corresponding to perturbations of both the amplitudes and phases of the element weights minimizing the relative weight perturbations $\sum_{n=1}^{N} (|\Delta w_n|/a_n)^2$ [Eq. (9)] and the absolute weight perturbations, $\sum_{n=1}^{N} |\Delta w_n|^2$, [Eq. (11)], respectively. For each of these two cases we show three graphs, the first with the perturbed pattern for θ from -90° to 90°, the second with the cancellation pattern from -90° to 90°, and the third with both the original and the perturbed patterns in the range from 0° to 25°.

We see that the cancellation beam in Figure 2b for minimized relative weight perturbations, which corresponds to a taper of the element excitations equal to the square of the original amplitude taper, has a much broader mainlobe and much lower sidelobes than the cancellation beam in Figure 3b which corresponds to a uniform distribution of the element amplitudes. This means that the original pattern is distorted more strongly in the vicinity of the null for minimized relative weight perturbations than it is for minimized total weight perturbations, and conversely that the original pattern is distorted less strongly far away from the null for the minimized relative weight perturbations than it is for minimized total weight perturbations. (In a mean square sense, as is demonstrated in Appendix A, the distortion of the original pattern is minimized by the minimized total weight perturbation scheme.) Note that the broad mainlobe of the cancellation pattern in Figure 2b, being out of phase with the fifth sidelobe of the original pattern, is in phase with the fourth and sixth sidelobes of the original pattern with the result that these sidelobe levels are raised (see Figures 2a, 2c). Indeed the effect of the mainlobe of this cancellation pattern extends even to the third and seventh sidelobes of the original pattern which, being out of phase with the mainlobe of the cancellation pattern, are lowered slightly from their original values. By the time the second or eighth sidelobes are reached, however, the effect of the cancellation pattern of Figure 2b is no longer felt and the perturbed and original patterns are virtually indistinguishable. In contrast, the cancellation beam of Figure 3b, which tapers much more slowly than does the cancellation beam of Figure 2b, can be seen by referring to Figure 3c to have a small effect as far as the first sidelobe of the original pattern.

11. Drane, C., Jr. (1964) Dolph-Chebyshev excitation coefficient approximation IEEE Trans. Antennas Propag., AP-12:781-782.

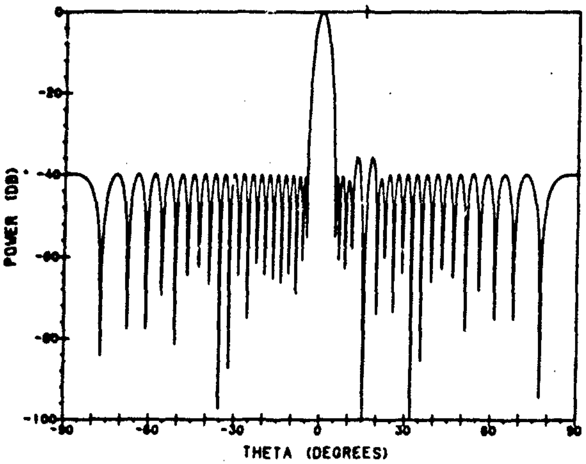


Figure 2a. Perturbed Pattern With One Null Imposed at 15.23° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. $\theta = -90°$ to $+90°$

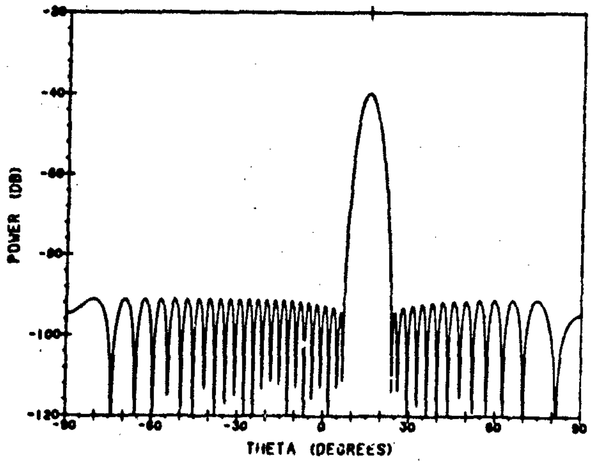


Figure 2b. Cancellation Patterns to Impose One Null at 15.23° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. $\theta = -90°$ to $+90°$

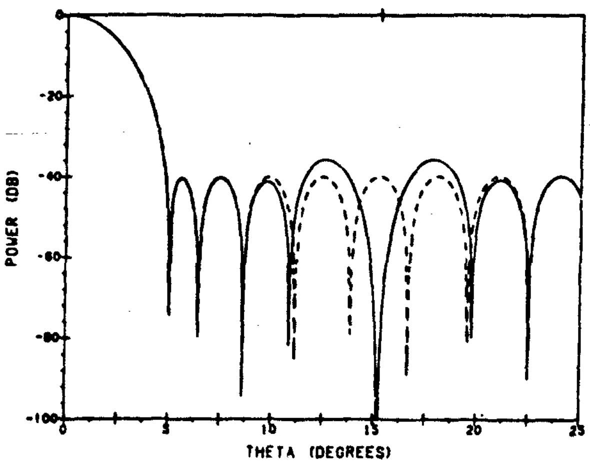


Figure 2c. Original Pattern and Perturbed Pattern With One Null Imposed at 15.23° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Pertubations. $\theta = 0°$ to $25°$

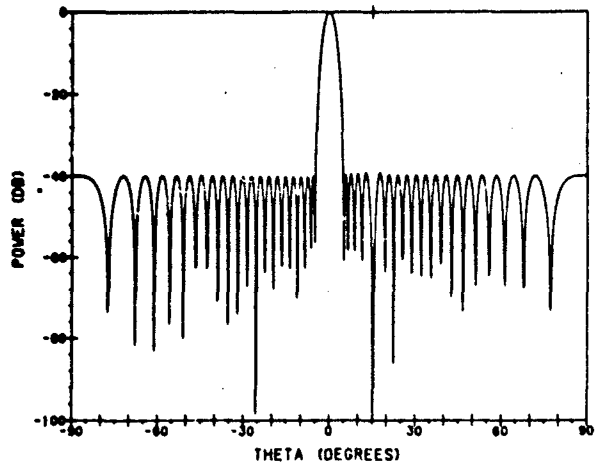
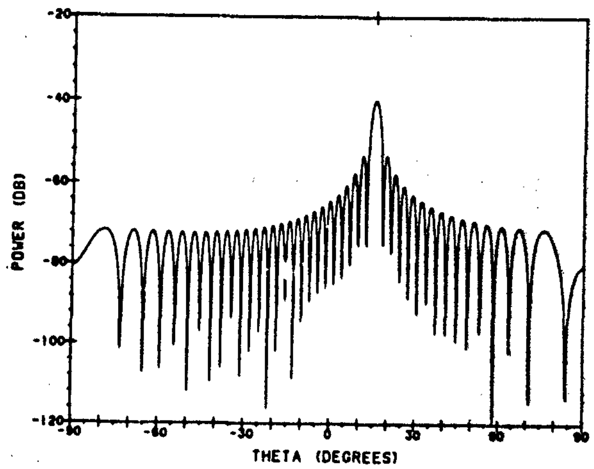


Figure 3a. Perturbed Pattern With One Null Imposed at 15.23° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = -90° to +90°

Figure 3b. Cancellation Pattern to Impose One Null at 15.23° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = -90° to +90°



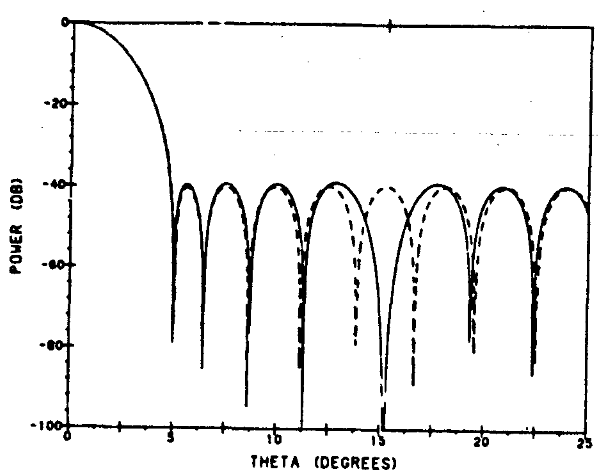


Figure 3c. Original Pattern and Perturbed Pattern With One Null Imposed at 15.23° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = 0° to 25°

Figures 4a-4c and 5a-5c show patterns for a single imposed null at 15.23° corresponding to perturbations of only the phases of the element weights. Figures 4a-4c show the results when the sum of the squares of the phase perturbations, $\sum_{n=1}^{N} \phi_n^2$ is minimized, [Eq. (22)], and Figures 5a-5c show the results when $\sum_{n=1}^{N} (a_n \phi_n)^2$ is minimized [Eq. (24)]. The principal change in these patterns from those in Figures 2a-2c and 3a-3c corresponding to perturbations of both the amplitudes and the phases of the weights is that now the cancellation pattern is the superposition of two beams, one with a peak in the direction of the null, and the other with a peak of opposite phase in the symmetric direction on the other side of broadside. Hence any change in the pattern on the null side of broadside ($\theta > 0$) is accompanied by an equal and opposite change on the other side of broadside ($\theta < 0$). Note especially that the sidelobe peak at -15.23° in Figures 4a and 5a is raised 6 dB. This is because the cancellation pattern to produce a null at +15.23° must have a magnitude equal to, and a phase opposite, that of the original pattern at 15.23° and hence adds in phase to the original pattern (which is symmetric around $\theta = 0$) at -15.23° doubling its magnitude there.

The cancellation pattern of Figure 4b is the superposition of two beams of the form of Figure 2b (that is, beams corresponding to an element amplitude taper equal to the square of the original taper), while the cancellation pattern of Figure 5b is the superposition of two beams of the form of Figure 3b (that is, beams corresponding to a uniform element amplitude distribution). On the null side of broadside, $\theta > 0$, the phase-only cancellation patterns are dominated by exactly the same beams that are used for cancellation in the corresponding patterns when both amplitudes and phases of the element excitations vary. Hence the basic features of the phase-only patterns for positive θ are the same as those noted above for combined amplitude and phase perturbations, Figures 4a-4c being similar to Figures 2a-2c and Figures 5a-5c being similar to Figures 3a-3c.

In Table 1 we have tabulated the values of the beam coefficients, and the values of the original and perturbed patterns at the null positions for this and succeeding cases. For one null at 15.23° and perturbations of both amplitudes and phase, it will be noted that the value of the beam coefficient equals the negative of the original pattern at the null position. This is a consequence of the normalization of the cancelling beams discussed at the end of the previous section. For phase-only perturbations, the beam coefficient corresponding to the minimization of the sum of the squares of the phase perturbations again equals the negative of the original pattern, but the beam coefficient corresponding to the minimization of $\sum_{n=1}^{N} (a_n \phi_n)^2$ is slightly larger. This deviation of the latter beam coefficient is a consequence of the fact that for phase-only perturbations and one imposed null, the beam directed at the null position must not only cancel the original pattern there but must also cancel the sidelobe of the beam directed at the symmetrical position on the other side of broadside. Since the sidelobes of the beams used for cancellation when $\sum_{n=1}^{N} \phi_n^2$ is minimized taper off considerably more quickly than the sidelobes of the beams used for cancellation when $\sum_{n=1}^{N} (a_n \phi_n)^2$ is minimized (compare Figures 2b with Figure 3b), the effect of the sidelobe of the beam directed at the symmetrical position of the null is more strongly felt in the latter case than it is in the former.

Note also that the depth of the nulls achieved when both the amplitudes and phases of the element coefficients are allowed to vary are considerably deeper than the null depths when only the phases are allowed to vary. This is a consequence of the small angle approximation (13) discussed in the previous section, used to linearize the phase-only nulling problem. The phase perturbations obtained by solving the linearized nulling problem do not give exact nulls when substituted back into the element weights. However, the smaller the phase perturbations, the better the approximation is. This explains why the null depth achieved when the sum of the squares of the phase perturbations is minimized is deeper than the null obtained when the weighted sum $\sum_{n=1}^{N} (a_n \phi_n)^2$ is minimized—the phase perturbations in the former case are in a least squares sense smaller than those in the latter case.

Following this discussion of imposing a single null, we now look at the case of two nulls imposed symmetrically at the -3 dB points of a sidelobe. Locations of $\theta = 14.54°$ and $15.94°$ were chosen corresponding to the -3 dB points of the fifth sidelobe of the unperturbed pattern. Figures 6a-6c and 7a-7c show the pattern corresponds to the perturbations of both the amplitudes and phases of the element excitations that minimize the sum of the squares of the relative weight perturbations and of the total weight perturbations respectively. Since the points at which the nulls are imposed belong to the same sidelobe and are approximately symmetrical with respect to the sidelobe maximum, the beam coefficients are approximately equal (see Table 1). The beams appear to add constructively over most of their range to form what has very much the appearance of a single beam (compare Figures 6b and 7b to Figures 2b and 3b respectively). The qualitative features of the patterns shown in Figures 6a-6c and 7a-7c are entirely similar to those of Figures 2a-2c and 3a-3c respectively.

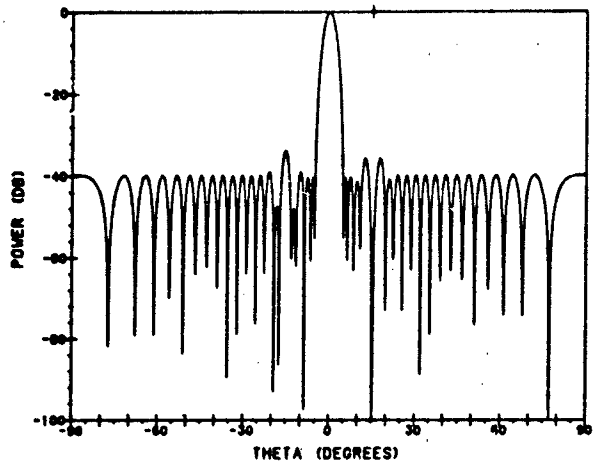
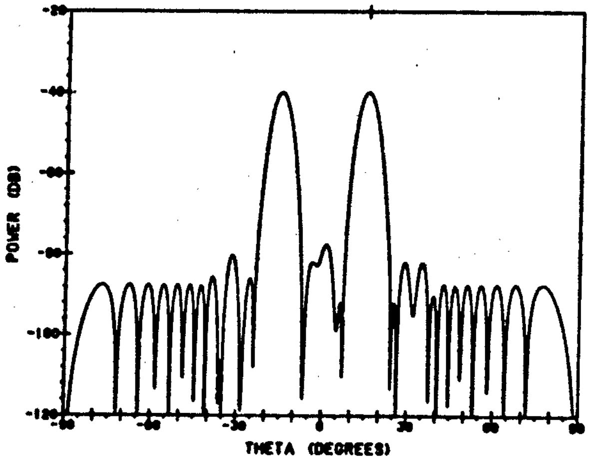


Figure 4a. Perturbed Pattern With One Null Imposed at 15.23° With Phase-Only Perturbations Minimizing $\Sigma \phi_n^2$. θ = -90° to +90°

Figure 4b. Cancellation Pattern to Impose One Null at 15.23° With Phase-Only Perturbations, Minimizing $\Sigma \phi_n^2$. θ = -90° to +90°



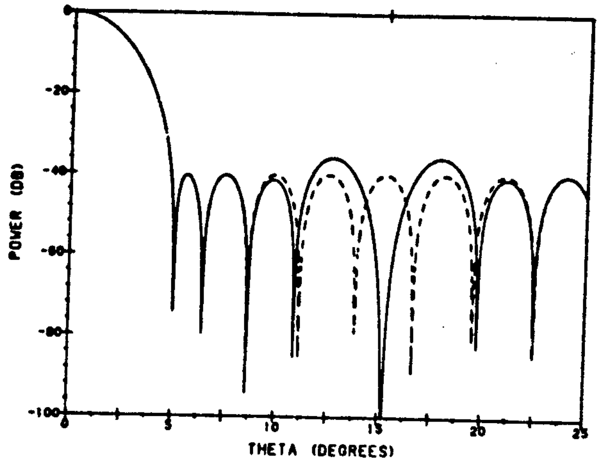


Figure 4c. Original Pattern and Perturbed Pattern With One Null Imposed at 15.23° With Phase-Only Perturbations, Minimizing $\Sigma \phi_n^2$. θ = 0° to 25°

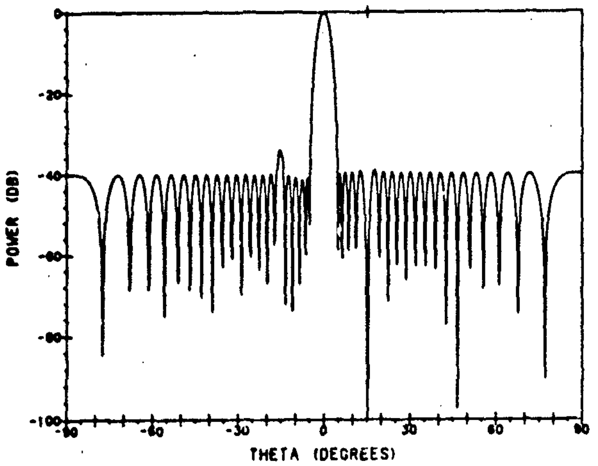
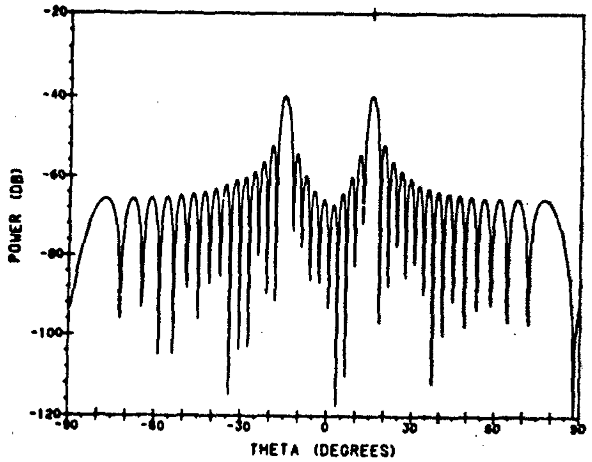


Figure 5a. Perturbed Pattern With One Null Imposed at 15.23° With Phase-Only Perturbations Minimizing $\Sigma(a_n \phi_n)^2$. θ = -90° to +90°

Figure 5b. Cancellation Pattern to Impose One Null at 15.23° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = -90° to +90°



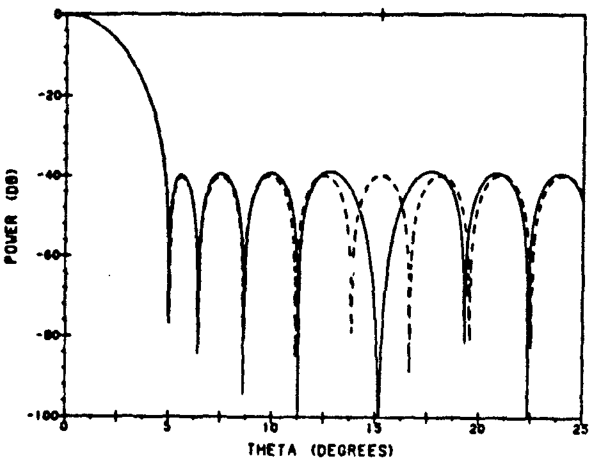


Figure 5c. Original Pattern and Perturbed Pattern With One Null Imposed at 15.23° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = 0° to 25°

Table 1. Values of the Imposed Null Locations (θ_{null}), Original Pattern (p_o) Beam Coefficients (b), and Perturbed Pattern (p) for Four Types of Weighted Perturbations. (Numbers in parentheses are powers of ten)

	p_o		$\sum(\lvert\Delta w_n\rvert/a_n)^2$ = min.		$\sum\lvert\Delta w_n\rvert^2$ = min.		$\sum\phi_n^2$ = min.		$\sum(a_n\phi_n)^2$ = min.	
θ_{null} (deg)	Value	dB	b	p(dB)	b	p(dB)	b	p(dB)	b	p(dB)
15.23	-0.101(-1)	-40	+0.101(-1)	< -260	+0.101(-1)	< -260	+0.101(-1)	-124	+0.104(-1)	-97
14.54	-0.721(-2)	-43	+0.411(-2)	< -260	+0.443(-2)	< -260	+0.420(-2)	-132	+0.480(-2)	-111
15.94	-0.714(-2)	-43	+0.351(-2)	< -260	+0.422(-2)	< -260	+0.343(-2)	-131	+0.399(-2)	-99
15.95	-0.714(-2)	-43	+0.601(-1)	< -260	+0.207(-1)	< -260	+0.604(-1)	-111	+0.218(-1)	-74
17.36	+0.716(-2)	-43	-0.601(-1)	< -260	-0.208(-1)	< -260	-0.605(-1)	-131	-0.224(-1)	-83
15.23	-0.101(-1)	-40	+0.516(-1)	< -260	+0.202(-1)	< -260	+0.525(-1)	-120	+0.216(-1)	-85
15.94	-0.714(-1)	-43	-0.428(-1)	< -260	-0.111(-1)	< -260	-C.437(-1)	-123	-0.122(-1)	-81

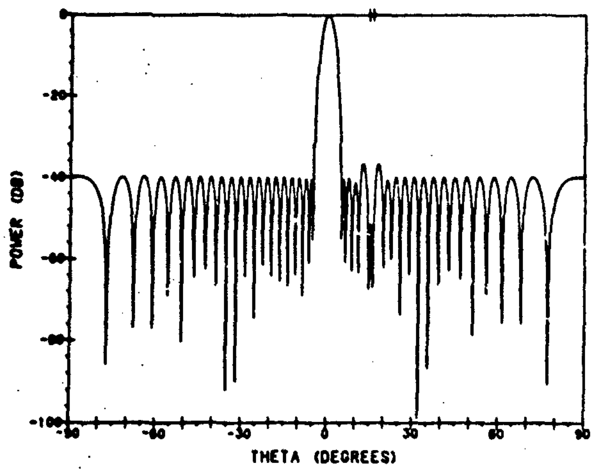


Figure 6a. Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = -90° to +90°

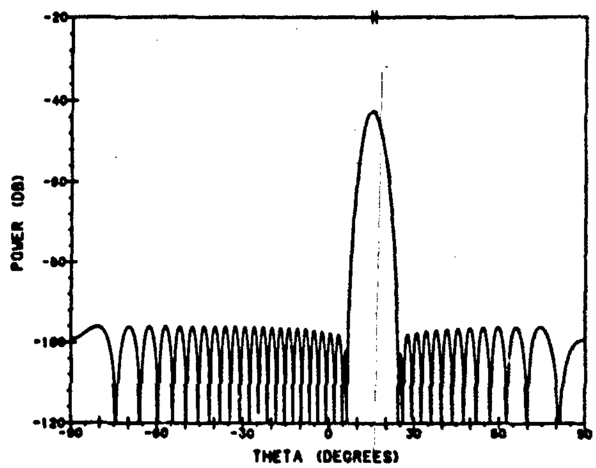


Figure 6b. Cancellation Pattern to Impose Two Nulls at 14.54° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = -90° to +90°

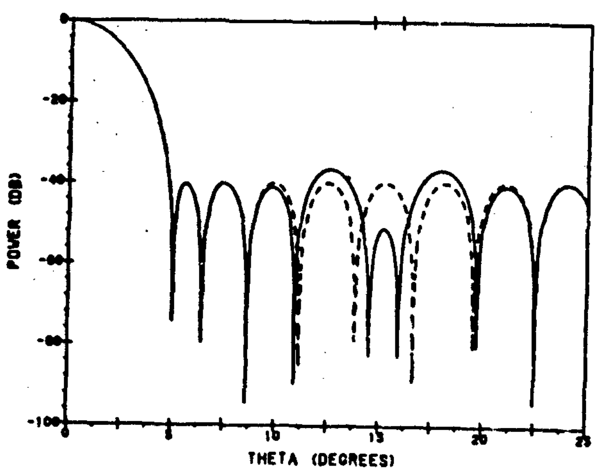


Figure 6c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = 0° to 25°

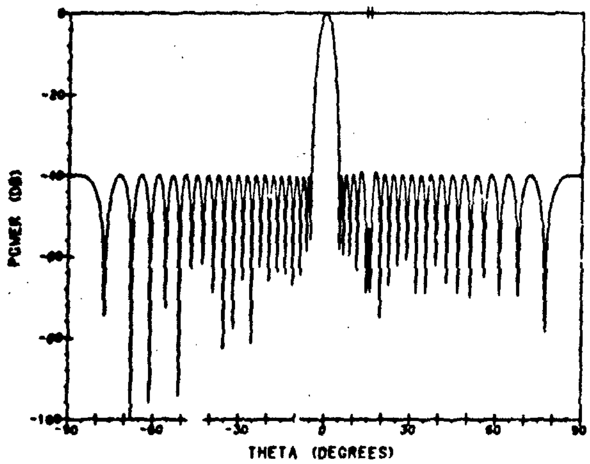
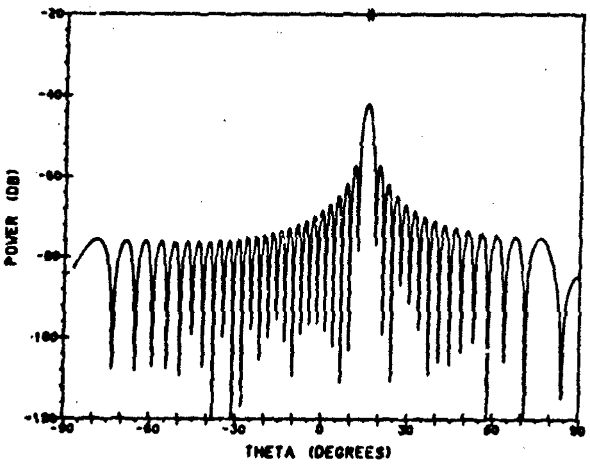


Figure 7a. **Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = -90° to +90°**

Figure 7b. **Cancellation Pattern to Impose Two Nulls at 14.54° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = -90° to +90°**



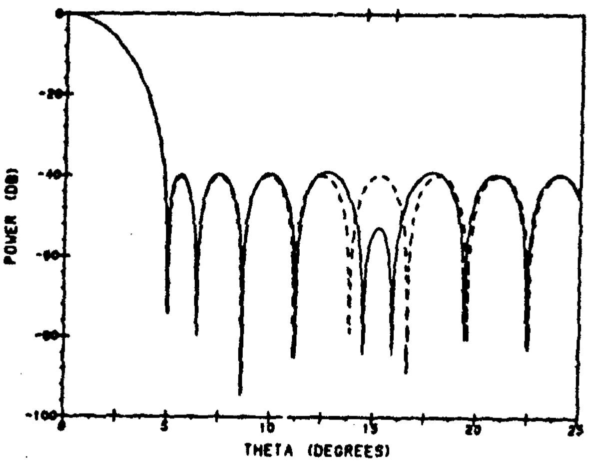


Figure 7c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = 0° to 25°

Figures 8a-8c and 9a-9c show the patterns obtained for the cases of the same two null locations at the -3 dB points of the fifth sidelobe when only perturbations of the element phases are allowed. Figures 8a-8c correspond to the minimization of the sum of the squares of the phase perturbations, and Figures 9a-9c to the minimization of $\sum_{n=1}^{N} (a_n \phi_n)^2$. Referring to the cancellation patterns shown in Figures 8b and 9b and comparing these with the cancellation patterns in Figures 4b and 5b respectively, we see that the two pairs of beams composing the phase-only cancellation patterns appear to add constructively to form one resultant beam pair. The qualitative features of the patterns shown in Figures 8a-8c and 9a-9c are thus the same as those noted above in the case of only one constrained null for Figures 4a-4c and 5a-5c respectively.

Referring to Table 1 we note that, unlike the case of one null, the beam coefficients for all four minimization types are considerably smaller than the corresponding values of the magnitude of the original pattern at the null locations. This of course is the result of the fact that the cancellation of the original pattern at the null locations is effected by the peak of the cancelling beam directed at one null position along with the mainlobe of the cancelling beam directed at the other null position so that the two beams help one another. This also explains why the beam coefficients obtained when $\sum_{n=1}^{N} (|\Delta w_n|/a_n)^2$ is minimized are smaller than those obtained when $\sum_{n=1}^{N} |\Delta w_n|^2$ is minimized. The mainlobe of the cancellation beams in the former case falls off considerably more slowly than does the mainlobe in the latter case (compare Figure 2b with Figure 3b) and so the peak of the beam directed at one null position can get more assistance in cancelling the original pattern from the mainlobe of the beam directed at the other null position. Some of the same features noted above for the case of one null are seen here as well. Combined amplitude and phase perturbations give deeper nulls than do phase perturbations alone, and minimization of $\sum_{n=1}^{N} \phi_n^2$ gives deeper nulls than does the minimization of $\sum_{n=1}^{N} (a_n \phi_n)^2$.

We next examine the case of two nulls imposed symmetrically on either side of a null of the original pattern. Locations of $\theta = 15.95°$ and $17.36°$ were chosen symmetrically placed on either side of the null between the fifth and sixth sidelobes of the original pattern and spaced apart the same distance (measured by $\sin\theta$) as the two nulls in the preceding case. Figures 10a-10c and 11a-11c show the patterns corresponding to the perturbations of both the amplitudes and phases of the element excitations that minimize respectively the sum of the squares of the relative weight and the total weight perturbations. Since the points at which the nulls are imposed belong to adjacent and almost equal sidelobes, and are approximately symmetrical with respect to the null between the sidelobes, the beam coefficients are approximately equal in magnitude but opposite in sign (see Table 1). Unlike the previous case where the mainlobes of the two component beams of the cancellation pattern added in phase to form a single mainlobe, here the cancellation pattern in the vicinity of the null positions is split into two lobes of opposite sign. The fact that the beam coefficients are of opposite sign means that each of the beams must do a considerable amount of work, as it were, to cancel the effect of the other beam, instead of being helped by the other beam as in the previous case. As a result, unlike the previous case, the beam coefficients are considerably larger in magnitude than the values of the original pattern at the null locations (see Table 1). This is especially the case for the beam coefficients corresponding to minimization of the relative weight perturbations which are almost nine times the magnitude of the original pattern values and three times as large as the beam coefficients corresponding to minimization of the total weight perturbations. The reason why the minimum relative perturbation beam coefficients are so much larger than the minimum total weight perturbation beam coefficients is that the taper of the mainlobe of the cancelling beams is much more gradual in the former case than in the latter. Hence each of the minimum relative weight perturbation beams must do correspondingly more work to cancel the mainlobe component of the other beam than is the case for the minimum total weight perturbation beams.

Note that even though the original pattern at the null locations is approximately -43 dB, the cancellation pattern corresponding to minimization of $\sum_{n=1}^{N} (|\Delta w_n|/a_n)^2$ rises to approximately -35 dB on either side of the pair of null locations, and the perturbed pattern rises to -32 dB. The location of the peaks of the cancellation pattern away from the pair of null locations (and the positions of the peaks of the component beams) may be explained as follows. Starting at, say, the left null position $\theta = 15.95°$ and moving to the left, the magnitude of both component beams decreases, but the magnitude of the beam pointed at $\theta = 15.95°$ decreases much less rapidly at first than does the magnitude of the beam pointed at $\theta = 17.36°$. Hence, the two beams being of opposite sign, the net effect is an increase in the cancellation pattern. A similar but less marked behavior is also seen in the case of minimization of the total weight perturbations (compare Figures 11a and 11c with Figures 10a and 10c respectively). The perturbation of the original pattern in the vicinity of the null positions is considerably more pronounced for minimization of the relative weight perturbations than it is for minimization of the total weight perturbations because the component beams of the cancellation pattern are much broader in the former case than in the latter (compare Figure 2b with Figure 3b).

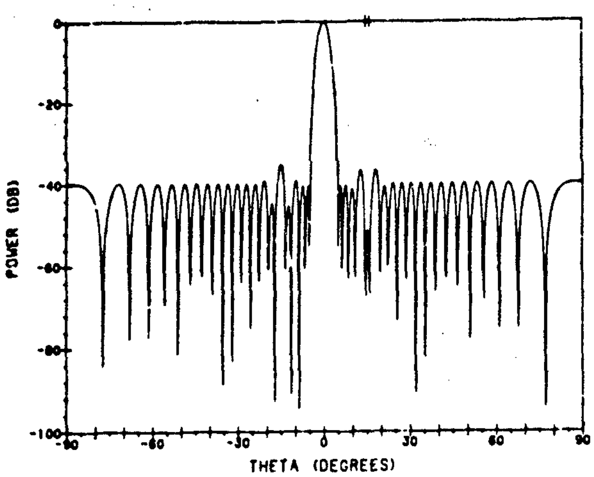


Figure 8a. Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^{\ 2}$. θ = -90° to +90°

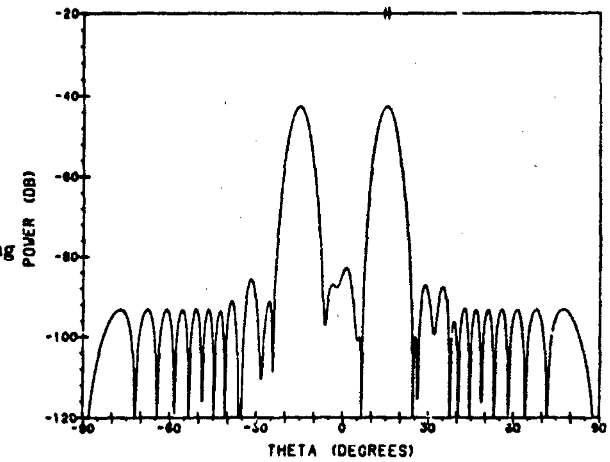


Figure 8b. Cancellation Pattern to Impose Two Nulls at 14.54° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^{\ 2}$. θ = -90° to +90°

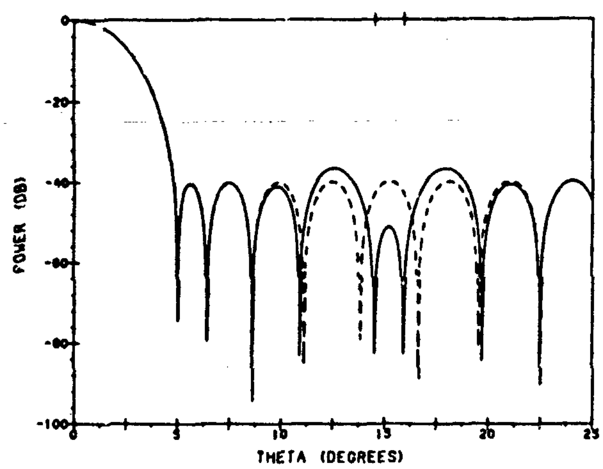


Figure 8c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^{\ 2}$. θ = 0° to 25°

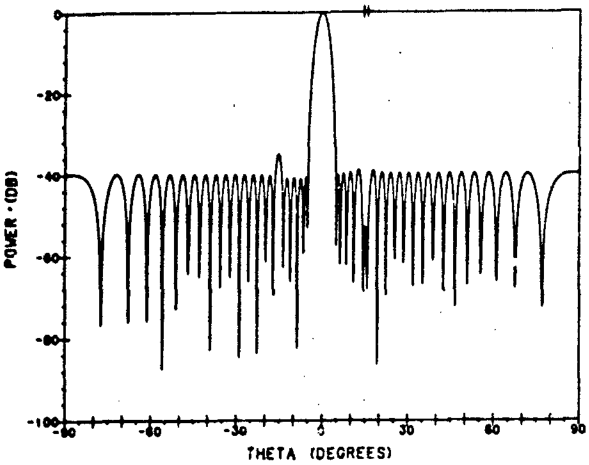


Figure 9a. Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = -90° to +90°

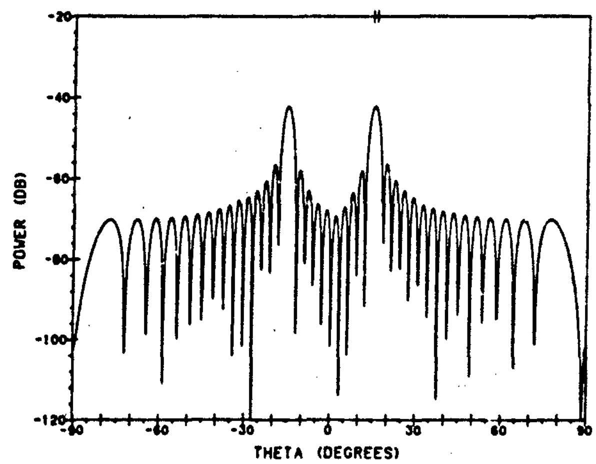


Figure 9b. Cancellation Pattern to Impose Two Nulls at 14.54° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = -90° to +90°

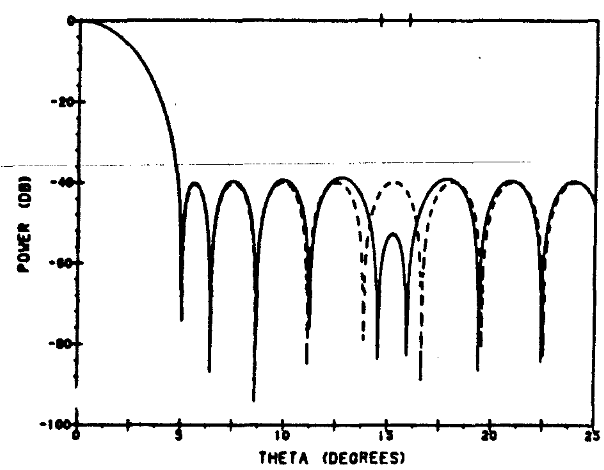


Figure 9c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 14.54° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = 0° to 25°

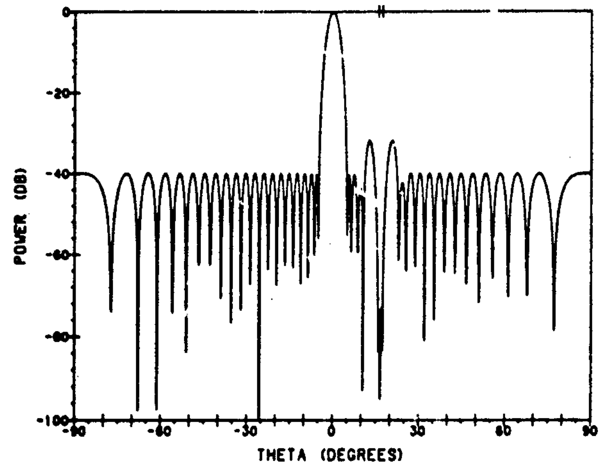
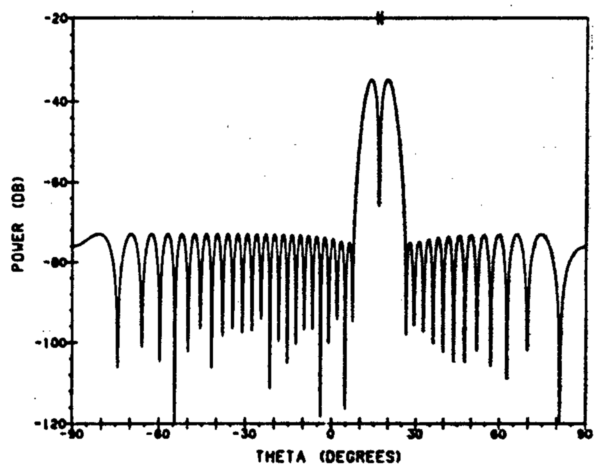


Figure 10a. Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = -90° to +90°

Figure 10b. Cancellation Pattern to Impose Two Nulls at 15.95° and 17.36° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = -90° to +90°



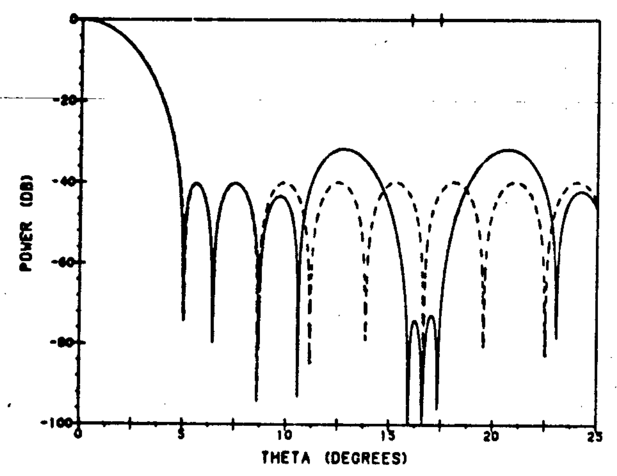


Figure 10c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = 0° to 25°

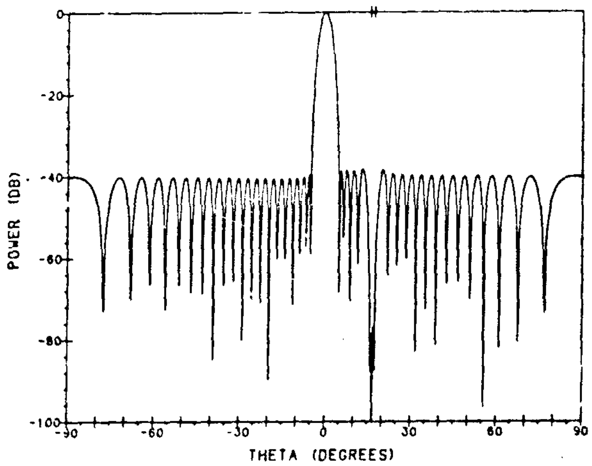


Figure 11a. Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = -90° to +90°

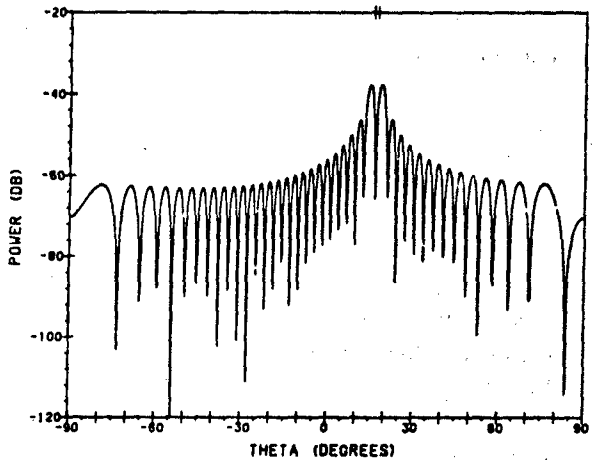


Figure 11b. Cancellation Pattern to Impose Two Nulls at 15.95° and 17.36° Combined Amplitude and Phase .rbations, Minimizing Total Weight r urbations. θ = -90° to +90°

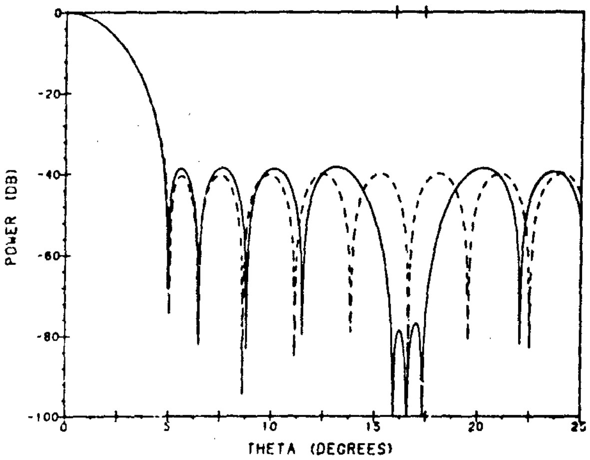


Figure 11c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = 0° to 25°

An additional consequence of the fact that the beam coefficients are much larger in magnitude than in the previous case of nulls on either side of a sidelobe maximum is that the cancellation patterns display considerably higher sidelobes than in the previous case (compare Figures 10b and 11b with Figures 6b and 7b respectively). The more gradual decrease of the sidelobes of the cancellation pattern is especially noticeable for the minimization of the total weight perturbations (compare Figure 11b with Figure 7b) so that the effect on the perturbed pattern is seen considerably further away from the null positions in this case than it is in the previous case of two nulls imposed symmetrically with respect to a sidelobe maximum (compare Figure 11c with Figure 7c).

Figures 12a-12c and 13a-13c show patterns obtained for the same two null locations, imposed symmetrically with respect to a null of the original pattern, when only the phases of the element weights are allowed to be perturbed. Figures 12a-12c correspond to minimizing the sum of the squares of the phase perturbations, and Figures 13a-13c to the minimization of $\sum_{n=1}^{N} (a_n \phi_n)^2$. The qualitative features of these patterns for $\theta > 0$ are very similar to those discussed above for combined amplitude and phase perturbations, Figures 12a-12c corresponding to Figures 10a-10c, and Figures 13a-13c to Figures 11a-11c. For $\theta < 0$, of course, the phase-only perturbation patterns are dominated by the pair of beams which are directed toward the symmetric positions of the null locations, and which have signs opposite to the corresponding cancellation beams directed at the null locations themselves.

Finally we examine the case of two nulls imposed asymmetrically within a sidelobe. Locations of $\theta = 15.23°$ and $15.94°$ were chosen corresponding respectively to the peak and right 3 dB point of the fifth sidelobe of the unperturbed pattern. Figures 14a-14c and 15a-15c show the patterns for combined amplitude and phase perturbations that minimize respectively the relative and total weight perturbations. In contrast to the case of two nulls imposed symmetrically with respect to a sidelobe maximum, here the mainlobe of the cancellation pattern for minimized relative weight perturbations is seen to be split into two lobes (compare Figure 14b with Figure 6b). Referring to Table 1 we see that the coefficients of the two beams composing the cancellation pattern are of opposite sign and have magnitudes more than five times larger than the corresponding unperturbed pattern values, indicating that the two beams are doing considerable work in cancelling the effect of each other in addition to cancelling the original pattern. This behavior of the cancellation pattern may be explained qualitatively as follows. The mainlobe of the beam directed at the location of the sidelobe peak at 15.23° is much broader than the sidelobe of the original pattern and hence tapers off much more gradually than does the unperturbed sidelobe. Hence, if the beam directed at 15.23° was exactly matched in magnitude to the original pattern there it would have a component considerably larger in magnitude than the value of the unperturbed pattern at 15.94°. To counter this unwanted "overshoot", as it were, of the beam directed at 15.23°, the beam directed at 15.94° must be of opposite sign. The component at 15.23° of the mainlobe of the beam directed at 15.94° in turn requires cancellation by the peak of the beam directed at 15.23° which forces the magnitude of the latter beam to exceed the magnitude of the original pattern at 15.23°.

A similar but less pronounced behavior of the cancellation pattern occurs for minimized total weight perturbations (compare Figure 15b with Figure 7b). The mainlobe of the beam directed at 15.23° in this case, being less wide than the mainlobe of the minimized relative weight perturbation beam but still twice as wide as the sidelobe of the original pattern, has a smaller "overshoot" at 15.94° requiring correspondingly less compensation by the beam directed at 15.94°.

Figures 16a-16c and 17a-17c show the patterns obtained for the case of the same two null locations at 15.23° and 15.94° when only the phases of the weights are allowed to vary. Figures 16a-16c correspond to minimization of $\sum_{n=1}^{N} \phi_n^2$ and Figures 17a-17c to the minimization of $\sum_{n=1}^{N} (a_n \phi_n)^2$. The behavior of these patterns for $\theta > 0$ is similar to that of the corresponding combined amplitude and phase perturbation patterns.

Following this discussion of the basic characteristics of nulling at selected points, we now discuss a set of computations performed to investigate cancellation in a given sector of a pattern. We define the power cancellation ratio in the interval $\Delta\theta = \theta_1 \le \theta \le \theta_2$

$$C = \frac{\max\limits_{\theta \epsilon \Delta\theta} \left[p(\theta) \right]^2}{\max\limits_{\theta \epsilon \Delta\theta} \left[p_o(\theta) \right]^2}$$

where $p_o(\theta)$ is the original pattern and $p(\theta)$ is the perturbed pattern.

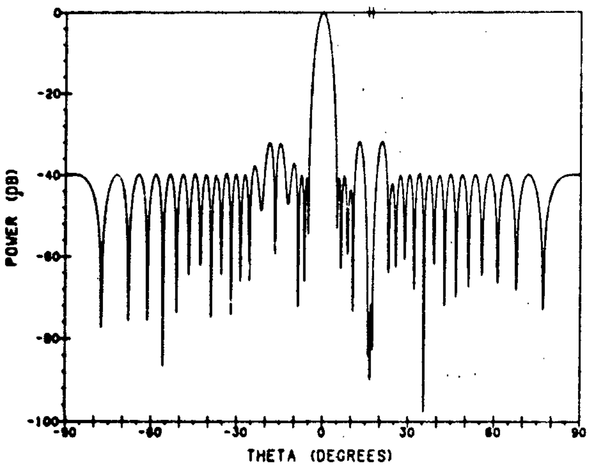


Figure 12a. Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^2$. θ = -90° to +90°

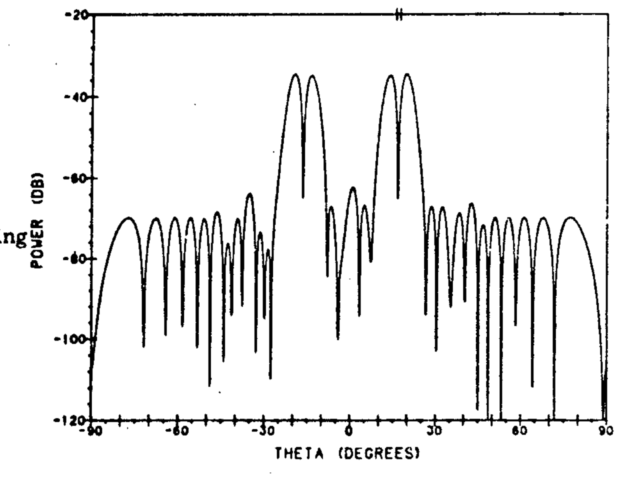


Figure 12b. Cancellation Pattern to Impose Two Nulls at 15.95° and 17.36° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^2$. θ = -90° to +90°

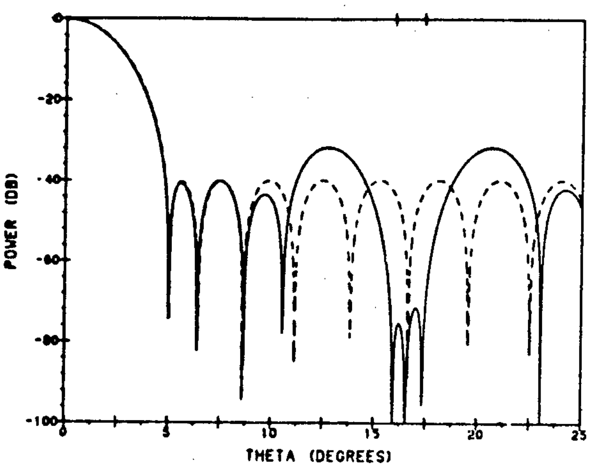


Figure 12c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^2$. θ = 0° to 25°

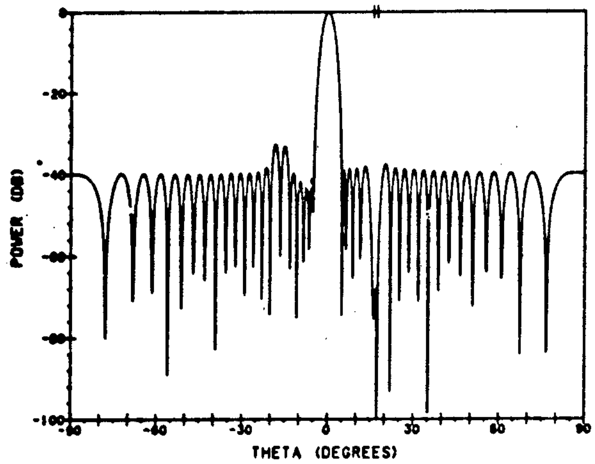


Figure 13a. Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = -90° to +90°

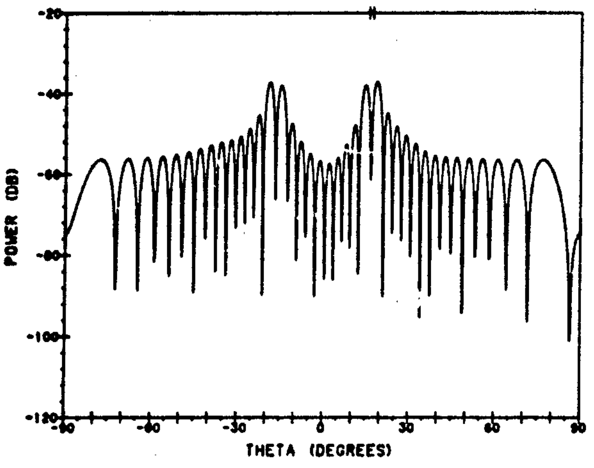


Figure 13b. Cancellation Pattern to Impose Two Nulls at 15.95° and 17.36° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = -90° to +90°

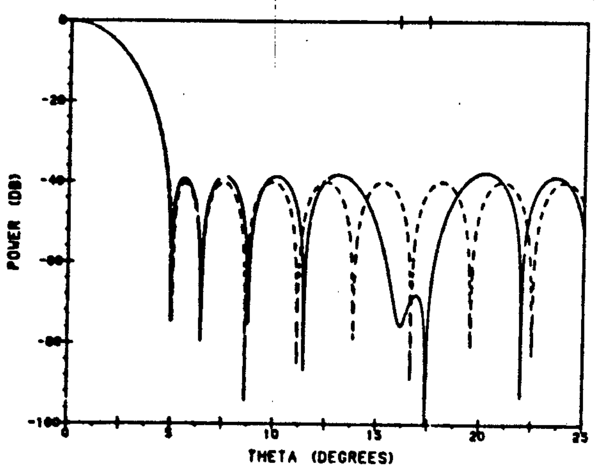


Figure 13c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.95° and 17.36° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = 0° to 25°

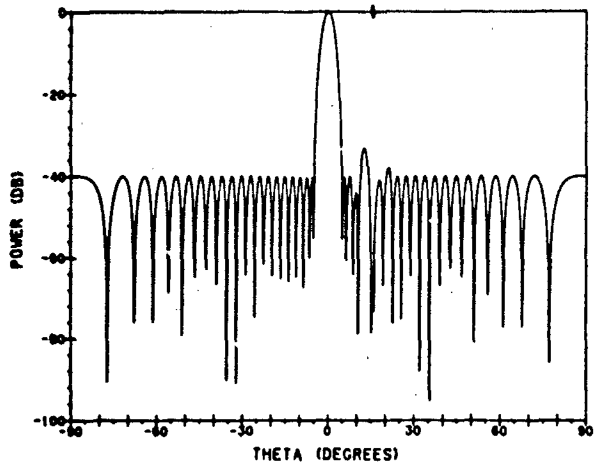


Figure 14a. Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = -90° to +90°

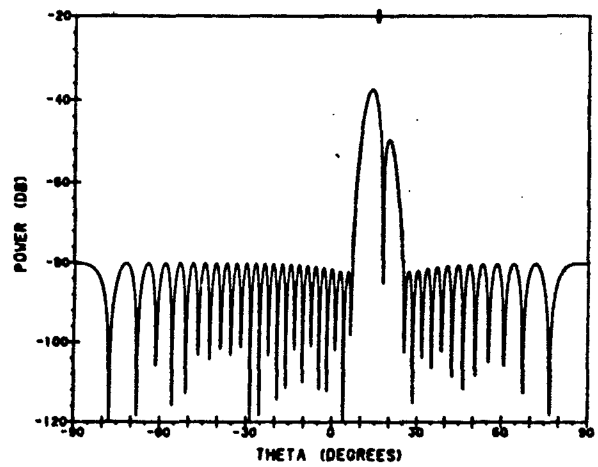


Figure 14b. Cancellation Pattern to Impose Two Nulls at 15.23° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = -90° to +90°

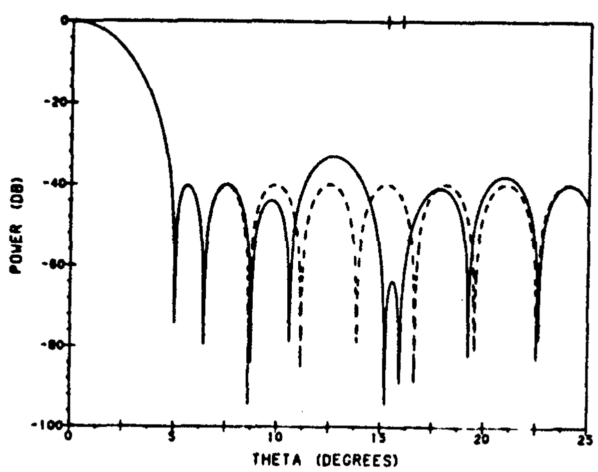


Figure 14c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Relative Weight Perturbations. θ = 0° to 25°

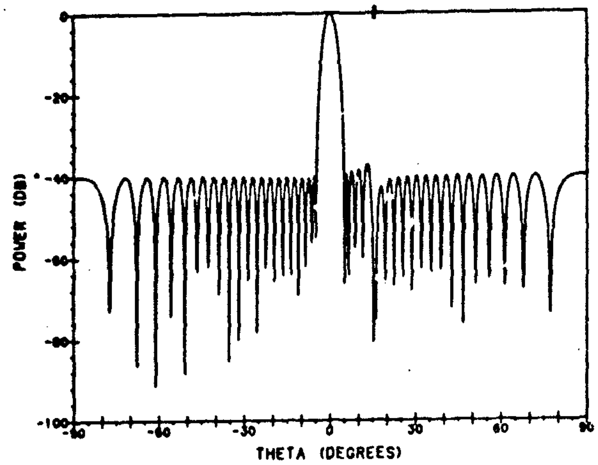


Figure 15a. Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.34° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = -90° to +90°

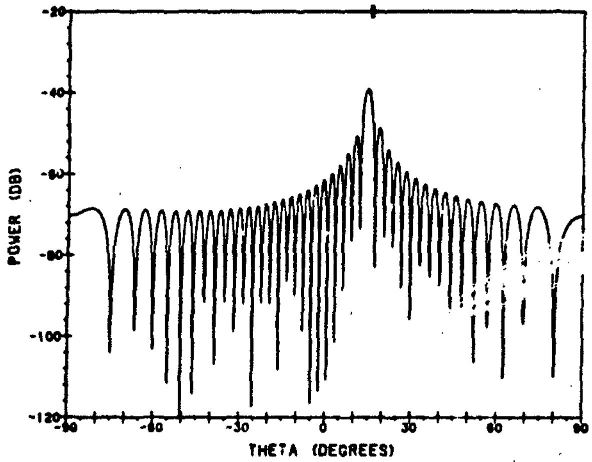


Figure 15b. Cancellation Pattern to Impose Two Nulls at 15.23° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = -90° to +90°

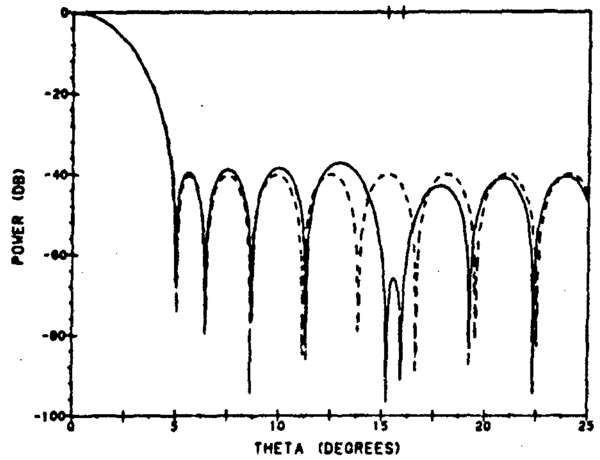


Figure 15c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.94° With Combined Amplitude and Phase Perturbations, Minimizing Total Weight Perturbations. θ = 0° to 25°

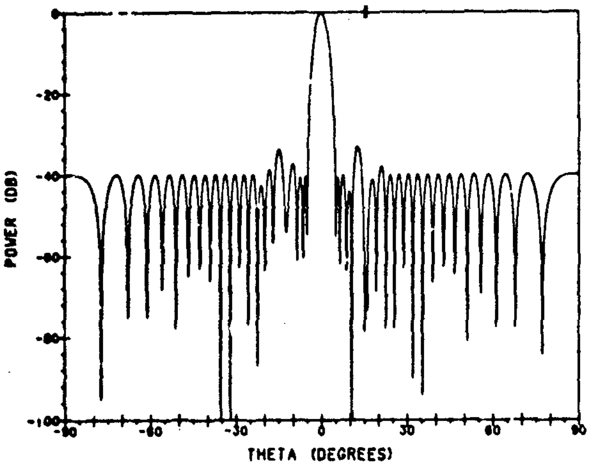


Figure 16a. Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^2$. θ = -90° to +90°

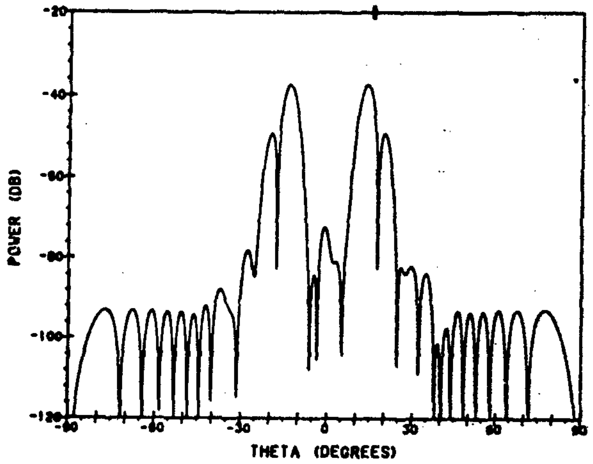


Figure 16b. Cancellation Pattern to Impose Two Nulls at 15.23° and 15.95° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^2$. θ = -90° to +90°

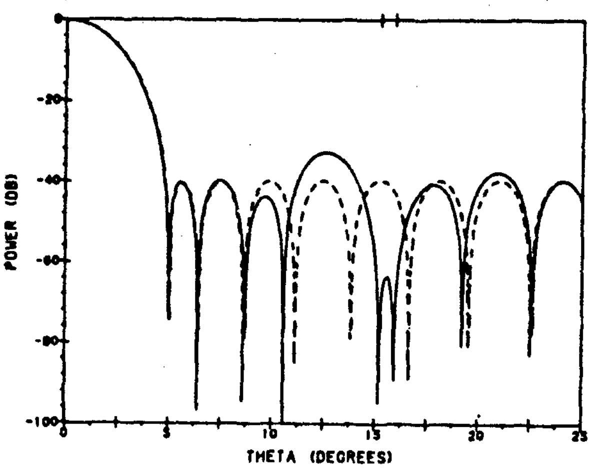


Figure 16c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma\phi_n^2$. θ = 0° to 25°

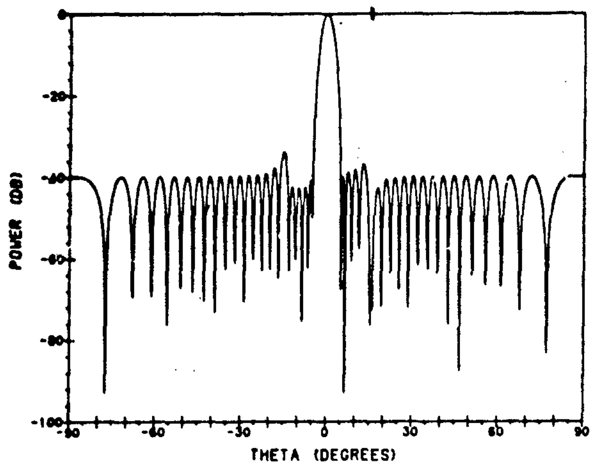


Figure 17a. Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n^2)$. θ = -90° to +90°

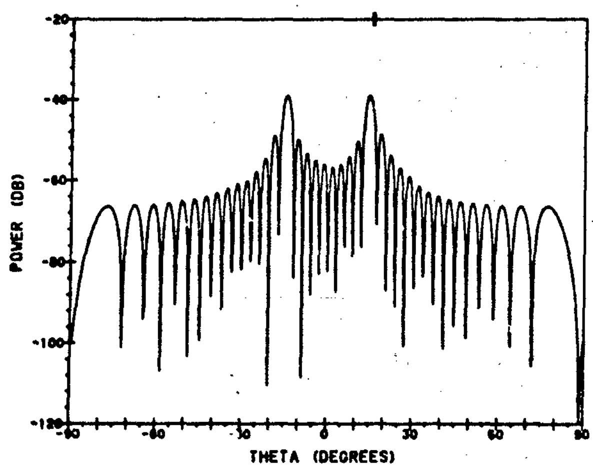


Figure 17b. Cancellation Pattern to Impose Two Nulls at 15.23° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = -90° to +90°

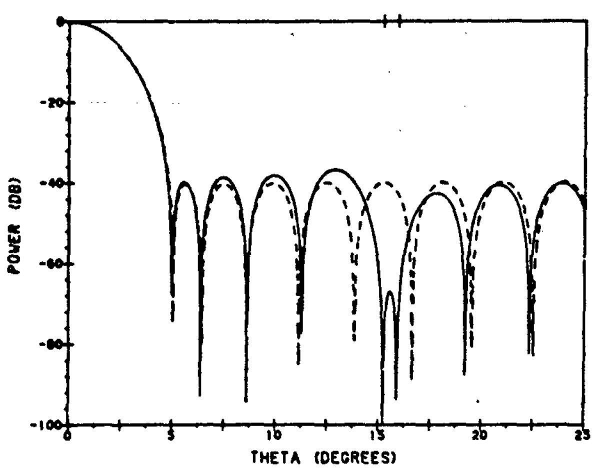


Figure 17c. Original Pattern and Perturbed Pattern With Two Nulls Imposed at 15.23° and 15.94° With Phase-Only Perturbations, Minimizing $\Sigma(a_n \phi_n)^2$. θ = 0° to 25°

The procedure used to study the behavior of the cancellation ratio for the four types of weight perturbations was as follows. Starting with the 41-element, 40 dB Chebyshev taper pattern, the left hand boundary of the sector of interest, θ_1, was fixed at the peak of the fifth sidelobe at $\theta = 15.23°$. A second location θ_2, was then found such that the cancellation ratio in the interval $[15.23°, \theta_2]$ was 30 dB when a null was imposed at 15.23° and θ_2 using combined amplitude and phase perturbations and requiring $\sum_{n=1}^{N} |\Delta w_n|^2$ to be minimized. This gave $\theta_2 = 15.78°$. The cancellation ratio for all four types of weight perturbations in the sector [15.23°, 15.78°] with nulls placed at 15.23° and 15.78° was then determined. Next the sector was enlarged to the right by adding a length (measured by $\sin\theta$) equal to $\sin(15.78°) - \sin(15.23°)$ giving $\theta_2 = 16.33°$. The cancellation ratio in the sector [15.23°, 16.33°] was then calculated for all four types of perturbations with nulls constrained to lie at 15.23°, 15.78°, and 16.33°. The procedure was then continued in the same way by adding equispaced nulls. The cancellation ratios obtained are shown in Table 2. Table 3 shows the values of beam coefficients and null depths for these cases.

Referring to Table 2, we see that the cancellation ratios obtained with the two types of combined amplitude and phase perturbations decrease steadily as the number of nulls increases indicating more effective cancellation. Slightly lower ratios are obtained by minimizing $\sum_{n=1}^{N} |\Delta w_n|^2$ than by minimizing $\sum_{n=1}^{N} (|\Delta w_n|/a_n)^2$. The improvement (that is, decrease) of the cancellation ratio with the increase in number of nulls may be explained by noting that in requiring nulls to be placed at various locations we are, equivalently, requiring the cancellation pattern to exactly match the original pattern at these locations. As the number of points at which the cancellation pattern is matched to the original pattern increases, the better is the overall fit of the cancellation pattern to the original pattern and hence the cancellation ratio decreases. Indeed, in the limit of using all the degrees of freedom at our disposal by imposing nulls at N locations, the cancellation pattern will exactly match the original pattern everywhere and hence the cancellation ratio will be minus infinity. The slightly lower cancellation ratios obtained by minimizing $\sum_{n=1}^{N} |\Delta w_n|^2$ as compared with those obtained by minimizing $\sum_{n=1}^{N} (|\Delta w_n|/a_n)^2$ is consistent with what we have observed above in our examination of the patterns corresponding to nulling at one or two selected points. The cancellation beams used when the relative weight perturbations are minimized have a much broader mainlobe than those used when the total weight perturbations are minimized, and hence are more likely to result in higher perturbed pattern sidelobe values in between the null locations.

Table 2. Values of the Cancellation Ratio (dB) for the Sector $15.23° \le \theta \le \theta_2$ With Equispaced Imposed Nulls

	Combined Amplitude and Phase Control		Phase Control Only	
θ_2 (deg)	$\sum(\lvert\Delta w_n\rvert/a_n)^2 = \min.$	$\sum\lvert\Delta w_n\rvert^2 = \min.$	$\sum\phi_n^2 = \min.$	$\sum(a_n\phi_n)^2 = \min.$
15.78	-27.8	-30.0	-27.8	-31.4
16.33	-43.5	-47.7	-45.5	-27.3
16.88	-54.8	-60.6	-51.8	-11.4
17.44	-62.5	-67.9	-29.5	-12.2
18.00	-85.3	-87.1	- 1.3	- 0.7

Table 3. Values of the Imposed Null Locations (θ_{null}), Original Pattern (p_o), Beam Coefficients (b), and Perturbed Pattern (p) for Four Types of Weight Perturbations. (Numbers in parentheses are powers of ten)

	p_o		Combined Amplitude and Phase Control				Phase-Only Control			
			$\sum_n \left(\lvert \Delta w_n \rvert / a_n \right)^2 = \min$		$\sum_n \lvert \Delta w_n \rvert^2 = \min$		$\sum_n \phi_n^2 = \min$		$\sum_n (a_n \phi_n)^2 = \min$	
θ_{null} (deg)	Value	dB	b	p (dB)	b	p (dB)	b	p (dB)	b	p (dB)
15.23	-0.10 (-1)	-40	+0.52 (-1)	< -260	+0.20 (-1)	< -260	+0.53 (-1)	-121	+0.22 (-1)	-87
15.78	-0.83 (-2)	-42	-0.43 (-1)	< -260	-0.11 (-1)	< -260	-0.44 (-1)	-123	-0.12 (-1)	-84
15.23	-0.10 (-1)	-40	-0.10 (+1)	< -260	-0.18 (0)	< -260	-0.93 (0)	-87	-0.17 (-1)	-67
15.78	-0.83 (-2)	-42	+0.21 (+1)	< -260	+0.37 (0)	< -260	+0.19 (+1)	-111	+0.36 (0)	-68
16.33	-0.35 (-2)	-49	-0.11 (+1)	< -260	-0.21 (0)	< -260	-0.10 (+1)	- 86	-0.21	-73
15.23	-0.10 (-1)	-40	-0.59 (+1)	< -260	-0.89 (0)	< -260	-0.75 (+1)	- 95	-0.13 (+1)	-54
15.78	-0.83 (-2)	-42	+0.17 (+2)	< -260	+0.24 (+1)	< -260	+0.21 (+2)	-108	+0.35 (+1)	-52
16.33	-0.35 (-2)	-49	-0.16 (+2)	< -260	-0.23 (+1)	< -260	-0.21 (+2)	- 95	-0.34 (+1)	-52
16.88	+0.26 (-2)	-52	+0.52 (+1)	< -260	+0.75 (0)	< -260	+0.68 (+1)	- 86	+0.11 (+1)	-56
15.23	-0.10 (-1)	-40	+0.54 (+1)	< -260	+0.62 (0)	< -260	-0.84 (-1)	- 72	-0.27 (0)	-53
15.78	-0.83 (-2)	-42	-0.27 (+2)	< -260	-0.33 (+1)	< -260	-0.70 (+1)	- 89	-0.26 (0)	-52
16.33	-0.35 (-2)	-49	+0.50 (+2)	< -260	+0.61 (+1)	< -260	+0.21 (+2)	- 76	+0.22 (+1)	-54
16.88	+0.26 (-2)	-52	-0.40 (+2)	< -260	-0.50 (+1)	< -260	-0.22 (+2)	- 71	-0.27 (+1)	-60
17.44	+0.78 (-2)	-42	+0.12 (+2)	< -260	+0.16 (+1)	< -260	+0.74 (+1)	- 70	+0.10 (+1)	-77
15.23	-0.10 (-1)	-40	+0.11 (+3)	< -260	+0.17 (+2)	< -260	+0.15 (+3)	- 42	+0.26 (+2)	-41
15.78	-0.83 (-2)	-42	-0.53 (+3)	< -260	-0.80 (+2)	< -260	-0.75 (+3)	- 41	-0.12 (+3)	-43
16.33	-0.35 (-2)	-49	+0.10 (+4)	< -260	+0.16 (+3)	< -260	+0.15 (+4)	- 43	+0.25 (+3)	-52
16.88	+0.26 (-2)	-52	-0.10 (+4)	< -260	-0.15 (+3)	< -260	-0.15 (+4)	- 46	-0.2[illegible] (+3)	-52
17.44	+0.78 (-2)	-42	+0.52 (+3)	< -260	+0.79 (+2)	< -260	+0.79 (+3)	- 51	+0.1[illegible] (+3)	-44
18.00	+0.10 (-1)	-42	-0.11 (+3)	< -260	-0.17 (+2)	< -260	-0.17 (+3)	- 63	-0.28 (+2)	-41

The behavior of the cancellation ratios obtained for the two types of phase-only perturbations is in marked contrast to the behavior of the cancellation ratios for the two types of combined amplitude and phase perturbations. The cancellation ratios obtained by minimizing $\sum_{n=1}^{N} \phi_n^2$ increase as the number of null locations increases beyond three, and those obtained by minimizing $\sum_{n=1}^{N} (a_n \phi_n)^2$ increase as the number of null locations increases beyond two. For six null locations there is virtually no effective cancellation by either scheme of phase-only perturbations. In general the cancellation ratios obtained by minimizing the sum of the squares of the phase perturbations themselves are lower than those obtained by minimizing the sum of the squares of the weighted phase perturbations. The reason for the much poorer cancellation obtained with the phase-only perturbation schemes as compared with the combined amplitude and phase perturbation schemes is the small angle approximation used to linearize the phase-only nulling problem. As the number of imposed nulls increases and coupling between the component beams of the phase-only cancellation patterns increases, the beam coefficients become larger and larger in magnitude (see Table 3) and consequently the phase perturbations increase in magnitude. For example, for two nulls at 15.23° and 15.78°, the largest phase perturbations obtained by minimizing $\sum_{n=1}^{N} \phi_n^2$ and $\sum_{n=1}^{N} (a_n \phi_n)^2$ are 1.55° and 5.47° respectively. In contrast, for six nulls the largest phase perturbation is 42.7° and 18 phase perturbations are greater than 10° for minimizing $\sum_{n=1}^{N} \phi_n^2$ while for minimizing $\sum_{n=1}^{N} (a_n \phi_n)^2$ the largest phase perturbation is 65° and 10 phase perturbations are greater than 10°. The larger the phase perturbations, the worse the small angle approximation. Hence the null depth achieved degrades (see Table 3) and consequently the overall sector cancellation is less effective. The fact that the small angle approximation becomes less valid as the phase perturbations grow, also explains why somewhat better cancellation ratios are obtained by minimizing $\sum_{n=1}^{N} \phi_n^2$ than by minimizing $\sum_{n=1}^{N} (a_n \phi_n)^2$. The phase perturbations obtained with the former scheme are, in a mean squares sense, lower than those obtained with the latter scheme and so satisfy the small angle approximation better.

4. GENERALIZATIONS

In Section 2 we saw that the process of determining weight perturbations to place nulls at a set of specified locations led to a representation of the cancellation pattern in terms of beams. Minimization of the sum of the squares of the weight perturbations relative to the original weights led to a representation of the cancellation pattern in terms of beams corresponding to an element amplitude taper equal to the square of the original taper, while minimization of the sum of the squares of the total weight perturbations led to a representation in terms of beams corresponding to a uniform amplitude taper. We also saw that the latter minimization problem is equivalent to minimizing the mean square difference of the cancellation pattern over a full period of the variable $u = kd\sin(\theta)$. It is natural to inquire whether there is a minimization of some property of the cancellation pattern that corresponds to the minimization of the sum of the squares of the relative weight perturbations. The object of this section is to answer this question and in so doing to generalize some of the results obtained in Section 2.

We begin by returning to the problem addressed in Section 2 of determining combined amplitude and phase perturbations to place nulls at specified locations in a given pattern. Noting the general relation between the weights, w_n, of the elements of an equispaced linear array and the pattern, p(u) of the array

$$\sum_{n=1}^{N} |w_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |p(u)|^2 \, du \,, \tag{27}$$

where

$$p(u) = \sum_{n=1}^{N} w_n e^{-jd_n u} \,,$$

$$d_n = \frac{N-1}{2} - (n-1)$$

and

$$u = kd\sin\theta$$

it follows that the sum of the squares of the total perturbations Δw_n equals the mean of the squared cancellation pattern, $\Delta p(u) = p(u) - p_o(u)$; that is,

$$\sum_{n=1}^{N} |\Delta w_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |p(u) - p_o(u)|^2 \, du \,. \tag{28}$$

Hence minimizing the sum of the squares of the total weight perturbations is equivalent to minimizing the mean squared cancellation pattern.

We now ask whether there is a similar pattern minimization that corresponds to the minimization of the sum of the squares of the weight perturbations relative to the original weights:

$$\sum_{n=1}^{N} \left| \frac{\Delta w_n}{a_n e^{j d_n u_s}} \right|^2 = \sum_{n=1}^{N} \left(\frac{|\Delta w_n|}{a_n} \right)^2 .$$

In fact there is, and the answer is given in terms of minimizing the mean square of the convolution of the cancellation pattern with another pattern. For in general, if

$$p_1(u) = \sum_{n=1}^{N} w_{1n} e^{-j d_n u}$$

and

$$p_2(u) = \sum_{n=1}^{N} w_{2n} e^{-j d_n u}$$

then it is easy to verify by direct substitution that the convolution of p_1 and p_2 defined by

$$[p_1 * p_2](u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_1(u') \, p_2(u - u') \, du' = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_1(u - u') \, p_2(u') \, du'$$

has the representation

$$[p_1 * p_2](u) = \sum_{n=1}^{N} w_{1n} w_{2n} e^{-j d_n u} .$$

Hence, from Eq. (27),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| [p_1 * p_2](u) \right|^2 du = \sum_{n=1}^{N} |w_{1n} w_{2n}|^2 . \tag{29}$$

It follows by letting $w_{1n} = \Delta w_n$ and $w_{2n} = 1/a_n$ that

$$\sum_{n=1}^{N} \left(\frac{|\Delta w_n|}{a_n} \right)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| [(p - p_o) * g](u) \right|^2 du$$

where

$$g(u) = \sum_{n=1}^{N} \frac{1}{a_n} e^{-j d_n u} .$$

Thus the minimization of the sum of the squares of the relative weight perturbation is equivalent to the minimization of the mean square of the convolution of the cancellation pattern with the pattern corresponding to element weights equal to the reciprocals of the amplitudes of the original weights.

Note that we can also recast the equivalence of minimizing the sum of the squares of the total weight perturbations with minimizing the mean squared cancellation pattern [Eq. (38)] by letting $w_{1n} = \Delta w_n$ and $w_{2n} = 1$ in Eq. (29) thereby obtaining

$$\sum_{n=1}^{N} |\Delta w_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |[(p - p_o) * g](u)|^2 \, du \tag{30}$$

where

$$g(u) = \sum_{n=1}^{N} e^{-jd_n u} = \frac{\sin\left(\frac{Nu}{2}\right)}{\sin\left(\frac{u}{2}\right)} . \tag{31}$$

Hence, minimizing the sum of the squares of the total weight perturbations is equivalent to minimizing the mean square of the convolution of the cancellation pattern with $\sin\left(\frac{Nu}{2}\right)/\sin\left(\frac{u}{2}\right)$. The result, Eq. (30), can also be obtained directly from Eq. (28) by noting [as is easily verified using Eq. (31)] that the convolution of an equispaced linear array pattern with the function $\sin\left(\frac{Nu}{2}\right)/\sin\left(\frac{u}{2}\right)$ is the array pattern itself so that

$$\left[(p - p_o) * \frac{\sin\left(\frac{Nu}{2}\right)}{\sin\left(\frac{u}{2}\right)}\right](u) = p(u) - p_o(u) .$$

We can now go further and observe that the two sums of squares of weight perturbations,

$$\sum_{n=1}^{N} |\Delta w_n|^2 \quad \text{and} \quad \sum_{n=1}^{N} \left(\frac{|\Delta w_n|}{a_n}\right)^2$$

whose minimization in conjunction with placing nulls at a set of specified locations led to the two forms of cancelling beams treated in this report, are special cases of a more general weighted sum of squares

$$\sum_{n=1}^{N} \left(\frac{|\Delta w_n|}{c_n}\right)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |[(p - p_o) * g]\,(u)|^2 \, du \tag{32}$$

where

$$g(u) = \sum_{n=1}^{N} \frac{1}{c_n} e^{-jd_n u} .$$

It is then reasonable to ask whether, given any set of non-zero real coefficients c_n we can find the set of perturbations Δw_n that will result in the nulls at a set of prescribed locations $u = u_k$, $k = 1, 2, \ldots, M$, and that will minimize

$$\sum_{n=1}^{N} \left(\frac{|\Delta w_n|}{c_n} \right)^2$$

or, equivalently, minimize

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |[(p - p_o) * g](u)|^2 \, du .$$

The condition for nulls in the perturbed pattern at $u = u_k$, $k = 1, 2, \ldots, M$, is

$$0 = p(u_k) = p_o(u_k) + \Delta p(u_k)$$

$$= p_o(u_k) + \sum_{n=1}^{N} \Delta w_n e^{-jd_n u_k}$$

or

$$\sum_{n=1}^{N} e^{-jd_n u_k} \Delta w_n = - p_o(u_k), \quad k = 1, 2, \ldots, M \tag{33}$$

so that we wish to find the set of weight perturbations Δw_n satisfying Eqs. (33) and minimizing

$$\sum_{n=1}^{N} (|\Delta w_n| / c_n)^2 .$$

The problem can be expre ̲ed in the form of finding the vector of weight perturbations

$$\underline{\Delta w} = [\Delta w_1, \Delta w_2, \ldots, \Delta w_N]^T$$

satisfying

$$A\,\underline{\Delta w} = \underline{y}$$

and minimizing

$$(\underline{\Delta w})^{\dagger}\, C\, \underline{\Delta w}$$

where A is the M × N matrix

$$A = \begin{bmatrix} e^{-jd_1 u_1} & e^{-jd_2 u_1} & \cdots & e^{-jd_N u_1} \\ \vdots & \vdots & & \vdots \\ e^{-jd_1 u_M} & e^{-jd_2 u_M} & \cdots & e^{-jd_N u_M} \end{bmatrix}$$

C is the N × N diagonal matrix

$$C = \begin{bmatrix} c_1^{-2} & & & \\ & c_2^{-2} & & \emptyset \\ & & \ddots & \\ \emptyset & & & c_N^{-2} \end{bmatrix}$$

and

$$\underline{y} = -\,[p_o(u_1),\ p_o(u_2),\ \ldots,\ p_o(u_M)]^T\ . \tag{34}$$

Using a result from the theory of generalized inverses,[10] the solution is given by

$$\underline{\Delta w} = C^{-1}A^{\dagger}\,(AC^{-1}A^{\dagger})^{-1}\,\underline{y}$$

so that $\underline{\Delta w}$ is a linear combination of the columns of $C^{-1}A^{\dagger}$ with coefficients from the vector

$$\underline{b} = (AC^{-1}A^{\dagger})^{-1}\,\underline{y}\ .$$

But

$$C^{-1}A^{\dagger} = \begin{bmatrix} c_1^2 e^{jd_1 u_1} & c_1^2 e^{jd_1 u_2} & \cdots & c_1^2 e^{jd_1 u_M} \\ c_2^2 e^{jd_2 u_1} & c_2^2 e^{jd_2 u_2} & \cdots & c_2^2 e^{jd_2 u_M} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_N^2 e^{jd_N u_1} & c_N^2 e^{jd_N u_2} & \cdots & c_N^2 c^{jd_N u_M} \end{bmatrix}$$

so that

$$\Delta w_n = c_n^2 \sum_{m=1}^{M} b_m e^{jd_n u_m}$$

and the perturbed pattern

$$p(u) = p_o(u) + \Delta p(u)$$

where

$$\Delta p(u) = \sum_{m=1}^{M} b_m \sum_{n=1}^{N} c_n^2 e^{-jd_n(u-u_m)} .$$

If the original pattern $p_o(u)$ is real as we assumed in Section 2, then the cancellation pattern must likewise be real. This is assured by assuming even symmetry of the weighting coefficients c_n with respect to the phase reference center; that is,

$$c_n = c_{N-n+1}, \quad n = 1, 2, \ldots, N .$$

Then the cancellation pattern is represented as the sum of M real beams, one pointed at each of the null locations, corresponding to an amplitude taper equal to the square of the weighting coefficients c_n. The two types of beams treated in Section 2 are special cases of this result. Minimum relative weight perturbations are obtained, as noted above, by letting $c_n = a_n$ which gives $c_n^2 = a_n^2$ and a representation of the cancellation pattern in terms of beams corresponding to an amplitude taper equal to the square of the original taper. Minimum total weight perturbations are obtained by letting $c_n = 1$ so that $c_n^2 = 1$ and the cancellation pattern is represented as the sum of beams corresponding to a uniform amplitude distribution (that is, $\sin\left(\frac{Nu}{2}\right)/\sin\left(\frac{u}{2}\right)$ beams). We could equally well consider other types of beams. For example, by letting $c_n = \sqrt{a_n}$, we obtain $c_n^2 = a_n$ so that the cancellation beams are replicas of the original pattern itself.

A decision regarding which type of beams to use for nulling in a particular situation must take into account the characteristics of the cancelling beams and how they affect the perturbations of the original pattern at points other than the imposed null locations. We saw in Section 3 that the deviations of the perturbed pattern from the original pattern at points other than the null locations depend strongly on the width of the mainlobe and the taper of the sidelobes of the cancelling beams. Because of the inverse relationship between width of mainlobe and taper of sidelobes, there is a tradeoff between relatively small perturbations spread out for a relatively large distance from the null locations (as with the $\sin\left(\frac{Nu}{2}\right)/\sin\left(\frac{u}{2}\right)$ beams), and relatively large perturbations restricted to a relatively small vicinity of the null locations (as with the beams corresponding to a taper proportional to the square of an already highly tapered element amplitude distribution). It is possible that the relationship established in this section between the type of cancelling beams and the minimization of the convolution of the cancellation pattern with a pattern uniquely associated with the type of cancelling beams, can be used to help clarify and quantify this tradeoff between weak, distributed perturbations and strong, localized perturbations, but more work remains to be done here.

Although we have so far in this section restricted our attention to combined amplitude and phase perturbations, the results established here can be extended in a parallel way to phase-only nulling as well. Equation (32) can be used as the starting point for an analysis of phase-only nulling, with the form of the weight perturbations given by

$$
\begin{aligned}
\Delta w_n &= w_n - w_{on} \\
&= w_{on} e^{j\phi_n} - w_{on} \\
&= w_{on} (e^{j\phi_n} - 1)
\end{aligned}
$$

where ϕ_n is the phase perturbation of the n^{th} weight. For small perturbations,

$$\Delta w_n \approx j\, w_{on} \phi_n$$

so that the generalized linearized phase-only nulling problem is to determine the set of ϕ_n that satisfy the equation system [similar to Eq. (33)]

$$\sum_{n=1}^{N} e^{-jd_n u_k} (j\, w_{on} \phi_n) = -\, p_o(u_k), \quad k = 1, 2, \ldots, M \tag{35}$$

and minimize

$$\sum_{n=1}^{N} \left(\frac{|w_{on}|\, \phi_n}{c_n} \right).$$

Splitting Eq. (35) into its real and imaginary parts we obtain

$$\left.\begin{aligned} &\sum_{n=1}^{N} a_n \phi_n \sin(d_n u_k - \psi_n) = -\, p_o(u_k) &&(36a)\\ &\sum_{n=1}^{N} a_n \phi_n \cos(d_n u_k - \psi_n) = 0 &&(36b) \end{aligned}\right\} \quad k = 1, 2, \ldots, M$$

where we have let $w_{on} = a_n e^{j\psi_n}$. The phase-only nulling problem can then be expressed in the form of finding the vector of phase perturbations

$$\underline{\phi} = [\phi_1, \phi_2, \ldots, \phi_N]^T$$

satisfying

$$A\underline{\phi} = \underline{y}$$

and minimizing

$$\underline{\phi}^T C \underline{\phi}$$

where A is the M × N matrix

$$A = \begin{bmatrix} a_1 \sin(d_1 u_1 - \psi_1) & a_2 \sin(d_2 u_1 - \psi_2) & \cdots & a_N \sin(d_N u_1 - \psi_N) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_1 \sin(d_1 u_M - \psi_1) & a_2 \sin(d_2 u_M - \psi_2) & \cdots & a_N \sin(d_N u_M - \psi_N) \end{bmatrix}.$$

C is the N × N diagonal matrix

$$C = \begin{bmatrix} (a_1/c_1)^2 & & & \emptyset \\ & (a_2/c_2)^2 & & \\ & & \ddots & \\ \emptyset & & & (a_N/c_N)^2 \end{bmatrix}$$

and $\underline{y}$ is given as before by Eq. (34). Then

$$\underline{\phi} = C^{-1} A^T (AC^{-1}A^T)^{-1} \underline{y}$$

so that $\underline{\phi}$ is a linear combination of the columns of $C^{-1}A^T$ with coefficients from the vector

$$\underline{b} = (AC^{-1}A^T)^{-1} \underline{y} .$$

Since

$$C^{-1}A^T = \begin{bmatrix} c_1^2 a_1^{-1} \sin(d_1 u_1 - \psi_1) & c_1^2 a_1^{-1} \sin(d_1 u_2 - \psi_1) & \cdots & c_1^2 a_1^{-1} \sin(d_1 u_M - \psi_1) \\ c_2^2 a_2^{-1} \sin(d_2 u_1 - \psi_2) & c_2^2 a_2^{-1} \sin(d_2 u_2 - \psi_2) & \cdots & c_2^2 a_2^{-1} \sin(d_2 u_M - \psi_2) \\ \vdots & \vdots & & \vdots \\ c_N^2 a_N^{-1} \sin(d_N u_1 - \psi_N) & c_N^2 a_N^{-1} \sin(d_N u_2 - \psi_N) & \cdots & c_N^2 a_N^{-1} \sin(d_N u_M - \psi_M) \end{bmatrix}$$

it follows that

$$\phi_n = c_n^2 a_n^{-1} \sum_{m=1}^{M} b_m \sin(d_n u_m - \psi_n) . \qquad (37)$$

If the original phases, ψ_n, are odd symmetric with respect to the phase reference center, then even symmetry of the weighting coefficients, c_n, assures that the ϕ_n are odd symmetric and hence that Eq. (36b) is automatically satisfied. The two special cases treated in Section 2, the minimization of $\sum \phi_n^2$ and $\sum (a_n \phi_n)^2$, correspond to the choice of $c_n = a_n$ and $c_n = 1$ respectively, for which Eq. (37) yields Eq. (19) and Eq. (23).

The cancellation pattern corresponding to Eq. (37) is

$$\Delta p(u) = \sum_{n=1}^{N} \Delta w_n \, e^{-jd_n u}$$

$$= \sum_{n=1}^{N} w_{on} \left(e^{j\phi_n} - 1\right) e^{-jd_n u}$$

$$\approx j \sum_{n=1}^{N} w_{on} \, \phi_n \, e^{-jd_n u}$$

$$= \sum_{m=1}^{M} b_m \sum_{n=1}^{N} \frac{c_n^2}{2} \left\{ e^{-jd_n(u-u_m)} - e^{-j[d_n(u+u_m) - 2\psi_m]} \right\} .$$

For $\psi_n = d_n u_s$, this gives the approximate representation of the phase-only cancellation pattern as the superposition of M pairs of beams, one member of the pair directed at a null location, and the other member, of opposite sign, directed at a location symmetric with respect to the mainlobe of the original pattern.

5. CONCLUSIONS

In this report, we have considered the problem of imposing nulls in the pattern of a linear array of equispaced, isotropic radiators subject to the condition that the perturbations of the element weights be minimized. In Section 2 we analyzed four forms of the problem: (1a) nulls imposed with combined amplitude and phase perturbations, minimizing the sum of the squares of the perturbations relative to the original weights; (1b) nulls imposed with combined amplitude and phase perturbations, minimizing the sum of the squares of the total weight perturbations; (2a) nulls imposed with phase perturbations only, minimizing the sum of the squares of the perturbations relative to the original weights; and (2b) nulls imposed with phase perturbations only, minimizing the sum of the squares of the total weight perturbations. The results are summarized in Table 4. Forms (1a) and (1b) can be solved exactly for the required weight perturbations. Form (1a) leads to a representation of the cancellation pattern as a superposition of beams, one directed at each imposed null location, each beam corresponding to an element taper proportional to the square of the original taper; while form (1b) leads to a representation of the cancellation pattern as a superposition of beams of the $\sin\left(\frac{Nu}{2}\right) / \sin\left(\frac{u}{2}\right)$ type.

Table 4. Summary of Cases Treated and Results Obtained

Nulling Method	Problem Solved	Equivalent Pattern Problem	Cancellation Beam Excitation	Cancellation Beam Shape	Comments
combined amplitude and phase perturbations	$p(u_1) = o$ $\left\lvert \frac{\Delta w_n}{w_{on}} \right\rvert^2 = \text{min.}$	$p(u_1) = o$ minimize convolution of cancellation pattern with "reciprocal amplitude" pattern	initial taper squared; linear phase	"tapered beam" at $u = u_1$	exact solution for any number of nulls
	$p(u_1) = o$ $\lvert \Delta w_n \rvert = \text{min.}$	$p(u_1) = o$ $\int_{-\pi}^{\pi} \lvert p - p_o \rvert^2 \, du = \text{min.}$	constant amplitude, linear phase	sinc beam at $u = u_1$	
phase only perturbations	$p(u_1) = o$ $\phi_n^2 = \text{min.}$	$p(u_1) = o$ minimize convolution of cancellation pattern with "reciprocal amplitude" pattern	pair of initial taper squared, linear phase distributions	pair of "tapered" beams at $u = \pm u_1$	approximate solution; deteriorates for large number of nulls or high sidelobes
	$p(u_1) = o$ $(\lvert w_{on} \rvert \phi_n)^2 = \text{min.}$	$p(u_1) = o$ $\int_{-\pi}^{\pi} \lvert p - p_o \rvert^2 \, du = \text{min.}$	pair of constant amplitude, linear phase distributions	pair of sinc beams at $u = \pm u_1$	

Forms (2a) and (2b) cannot be solved exactly, but approximate solutions can be obtained by assuming that the required phase perturbations are small. Minimization of the relative weight perturbations is then approximated by the minimization of the sum of the squares of the phase perturbations, while minimizing the total weight perturbations is approximated by minimizing the sum of the squares of the products of the phase perturbations with the amplitudes of the weights. The resulting cancellation patterns can then be approximately represented as the sum of pairs of beams, one pair for each null location. One member of the pair of beams is directed at the null location, while the other, of opposite sign, is directed at the location symmetric with respect to the direction of the mainlobe axis of the original pattern. The form of beams is the same as that for the respective combined amplitude and phase perturbation problem; that is, beams corresponding to a taper equal to the square of the original amplitude taper for (2a) and $\sin\left(\frac{Nu}{2}\right)/\sin\left(\frac{u}{2}\right)$ beams for (2b).

In Section 3 we presented and discussed numerical calculations performed to display the principal features of the solutions obtained in Section 2. The original pattern was taken to be that of a 41 element array with half wavelength spacing and a 40 dB Chebyshev taper. We examined the patterns obtained with the four methods of Section 2 for the cases of one imposed null at the peak of a sidelobe, two nulls imposed symmetrically at the -3 dB points of a sidelobe, two nulls imposed symmetrically on either side of a null of the original pattern, and two nulls imposed asymmetrically, one at the peak of a sidelobe and the other at a -3 dB point of the sidelobe. We saw that in general the "tapered" beams corresponding to forms (1a) and (2a) resulted in larger perturbations of the original pattern in the vicinity of the imposed null locations and smaller perturbations far away from the imposed nulls than did the "sinc" beams corresponding to forms (1b) and (2b). This was a consequence of the fact that the "tapered" beams had wider mainlobes and lower sidelobes than did the "sinc" beams. Nulling in the phase-only patterns was accompanied by an approximately equal and opposite distortion of the original pattern at the points symmetric to the imposed null locations with respect to the axis of the original pattern mainlobe. As a result, both amplitude and phase weighting are required to produce nulls that are symmetric with respect to the direction of the mainlobe.

In Section 3 we also discussed the results of calculations performed to examine cancellation within a sector of the pattern as the width of the sector was increased by adding equispaced imposed nulls. We saw that the cancellation achieved by the two combined-amplitude-and-phase perturbation methods became increasingly effective as the width of the sector was increased, but that the depth of nulls and cancellation achieved by the two phase-only methods deteriorated severely as the width of the sector was increased. This deterioration of the performance of the phase-only methods was attributed to the fact that as imposed nulls were added, interference between the cancelling beams became more pronounced, the required phase perturbations were larger, and consequently the small angle approximation used to derive the phase-only solutions became more and more inaccurate

In Section 4 we considered the problem of finding a pattern minimization that was equivalent to minimizing the relative weight perturbations, in the same way that minimizing the mean square cancellation pattern was equivalent to minimizing the total weight perturbations. We found that there was an equivalent pattern minimization, namely, minimizing the mean square of the convolution of the cancellation pattern with the pattern corresponding to element weights equal to the reciprocals of the original amplitudes. We then showed that the types of minimization considered in Section 2 were special cases of a more general minimization of the sum of squares of the weight perturbations divided by arbitrary non-zero real coefficients—a minimization which had its equivalent counterpart in pattern space in terms of minimizing the mean square of the convolution of the cancellation pattern with the pattern corresponding to element weights equal to the dividing coefficients. We then solved this generalized minimization problem for combined amplitude and phase perturbations, and for phase-only perturbations.

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Appendix A

Nulling With Minimum Pattern Perturbations

In this appendix we solve the following problem:
Given a linear array of N equispaced isotropic radiators and an initial set of complex element weights, w_{on}, $n = 1, 2, \ldots, N$, determine the set of complex weights w_n, $n = 1, 2, \ldots, N$, which gives the array pattern with nulls at a set of M prescribed pattern locations, u_k, $k = 1, 2, \ldots, M$, $M < N$, and which differs as little as possible in the mean sense from the initial pattern. When we use the word "mean" here we define the interval over which the mean is taken to be the visible region $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ so that the variable $u = kd \sin\theta$ is to be integrated from -kd to +kd.

The initial pattern $p_o(u)$ is given in Section 2, as

$$p_o(u) = \sum_{n=1}^{N} w_{on}\, e^{-jd_n u}$$

and the perturbed pattern p(u) by

$$p(u) = \sum_{n=1}^{N} w_n\, e^{-jd_n u}.$$

The difference between the patterns is then

$$p(u) - p_o(u) = \sum_{n=1}^{N} (w_n - w_{on})\, e^{-jd_n u}$$

and the mean square difference between the patterns is

$$\frac{1}{2kd}\int_{-kd}^{kd} |p(u) - p_o(u)|^2 \, du = \sum_{m=1}^{N}\sum_{n=1}^{N} (w_m^* - w_{om}^*)(w_n - w_{on})$$

$$\bullet \frac{1}{2kd}\int_{-kd}^{kd} e^{-j(d_n - d_m)u} \, du$$

$$= \sum_{m=1}^{N}\sum_{n=1}^{N} (w_m^* - w_{om}^*)(w_n - w_{on}) \operatorname{sinc}[(d_n - d_m)kd]$$

$$= (\underline{\Delta w})^{\dagger} C (\underline{\Delta w})$$

where

$$\underline{\Delta w} = [w_1 - w_{o1}, w_2 - w_{o2}, \ldots w_N - w_{oN}]^T$$

$$= [\Delta w_1, \Delta w_2, \ldots, \Delta w_N]^T$$

and

$$\left[C\right]_{mn} = \operatorname{sinc}[(d_n - d_m)kd] = \left[C\right]_{nm} .$$

Note that for half wavelength spacing, $kd = \pi$ and C becomes the identity matrix so that

$$\frac{1}{2kd}\int_{-kd}^{kd} |p(u) - p_o(u)|^2 = (\Delta w)^{\dagger} (\Delta w)$$

$$= \sum_{n=1}^{N} |\Delta w_n|^2 . \qquad \text{(A1)}$$

The perturbed pattern is required to have nulls at the M locations $u = u_k$, $k = 1, 2, \ldots, M$, so that

$$0 = \sum_{n=1}^{N} w_n e^{-jd_n u_k}$$

$$= \sum_{n=1}^{N} [w_{on} + \Delta w_n] e^{-jd_n u_k}$$

or

$$\sum_{n=1}^{N} e^{-jd_n u_k} \Delta w_n = -p_o(u_k), \; k = 1, 2, \ldots, M .$$

In matrix form, the equation system for the M nulls is

$$A \, \underline{\Delta w} = \underline{y} \tag{A2}$$

where

$$A = \begin{bmatrix} e^{-jd_1 u_1} & e^{-jd_2 u_1} & \cdots & e^{-jd_N u_1} \\ e^{-jd_1 u_2} & e^{-jd_2 u_2} & \cdots & e^{-jd_N u_2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ e^{-jd_1 u_M} & e^{-jd_2 u_M} & \cdots & e^{-jd_N u_M} \end{bmatrix}$$

and

$$\underline{y} = -[p_o(u_1), \, p_o(u_2), \, \ldots, \, p_o(u_m)] .$$

The problem then is to find the solution to Eq. (A2) which minimizes $(\Delta w)^{\dagger} C (\Delta w)$.

Now the quadratic form $(\Delta w)^{\dagger} C (\Delta w)$ is positive definite since it represents the integral of a real power density. Hence, from the theory of generalized inverses[10] the desired solution is

$$\underline{\Delta w} \quad C^{-1} A^{\dagger} (A C^{-1} A^{\dagger})^{-1} \underline{y} . \tag{A3}$$

Equation (A3) gives the change in element weights for any spacing of the elements. For the special case of half-wavelength spacing we have already noted that the matrix C reduces to the identity matrix. Then

$$\underline{\Delta w} = A^{\dagger} (AA^{\dagger})^{-1} \underline{y} .$$

Hence $\underline{\Delta w}$ is a linear combination of the columns of A with coefficients from the vector $\underline{b} = (AA^{\dagger})^{-1} \underline{y}$. But the columns of A are the M vectors

$$[e^{jd_1 u_k}, e^{jd_2 u_k}, \ldots, e^{jd_N u_k}]^T \quad k = 1, 2, \ldots, M$$

so that

$$\Delta w_n = \sum_{k=1}^{M} b_k e^{jd_n u_k} .$$

This expression for the weight perturbations is identical to Eq. (10) obtained by requiring $\sum |\Delta w_n|^2$ to be a minimum. Of course this conclusion is also an immediate consequence of Eq. (A1), but it is of interest to obtain it by solving the minimum pattern perturbation problem with arbitrary uniform element spacing.

In the case of phase-only element control, half wavelength spacing, and small phase perturbations, starting with Eq. (A1) we have

$$\frac{1}{2kd} \int_{-kd}^{kd} |p(u) - p_o(u)|^2 \, du = \sum_{n=1}^{N} |\Delta w_n|^2$$

$$= \sum_{n=1}^{N} a_n^2 |e^{j\phi_n} - 1|^2$$

$$= \sum_{n=1}^{N} (a_n \phi_n)^2$$

so that minimizing $\sum (a_n \phi_n)^2$ is equivalent to minimizing the mean square pattern perturbation.

In the above, as mentioned earlier, we have defined the mean with respect to the visible region of the array pattern. Since the array pattern for an equispaced linear array is periodic with period 2π in the variable $u = kd \sin\theta$, it is also possible to define the mean with respect to one period of the variable u regardless of whether or not the visible region constitutes one complete period. If this is done, then the mean square difference between the original and perturbed pattern is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |p(u) - p_o(u)|^2 \, du = \frac{1}{2\pi} \sum_{m=1}^{N} \sum_{n=1}^{N} (w_m^* - w_{om}^*)(w_n - w_{on}) \int_{-\pi}^{\pi} e^{-j(d_n - d_m)u} \, du$$

$$= \frac{1}{2\pi} \sum_{m=1}^{N} \sum_{n=1}^{N} (w_m^* - w_{om}^*)(w_n - w_{on}) \frac{2 \sin[(n-m)\pi]}{(n-m)}$$

$$= \sum_{n=1}^{N} |\Delta w_n|^2$$

so that the sum of the squares of the weight perturbations equals the mean square of the cancellation pattern, the mean being taken with respect to a full period of the variable $u = kd \sin\theta$. Thus a least mean square match to an initial element excitation is equivalent to a least mean square match over one period of the initial pattern. In the special case of half wavelength spacing, $d = \lambda/2$, the period $-\pi \leq kd \sin\theta \leq \pi$ coincides with the visible region and Eq. (A4) becomes identical with Eq. (A1). When $d > \lambda/2$, then a match over a full period of the variable u implies a match over an angular sector smaller than the visible region. This however, is not too serious since the pattern is periodic outside this sector.